MAT 127: Calculus C, Spring 2015
Solutions to Some HW3 Problems

Below you will find detailed solutions to three problems from HW3. Since they were part of the WebAssign portion of this homework, your versions of these problems likely had different numerical coefficients. However, the principles behind the solutions and their structure are as described below.

Section 7.3, Problem 3 (webassign)

Find the general solution to the differential equation

$$(x^2+1)y' = xy. \tag{1}$$

This is a separable equation. Write $y' = \mathrm{d}y/\mathrm{d}x$, move everything involving y to LHS and everything involving x to RHS, and integrate:

$$\begin{aligned}
(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x} = xy \qquad &\Longleftrightarrow \qquad \frac{\mathrm{d}y}{y} = \frac{x}{x^2+1}\mathrm{d}x \\
&\Longleftrightarrow \qquad \int \frac{\mathrm{d}y}{y} = \int \frac{x}{x^2+1}\mathrm{d}x \\
&\Longleftrightarrow \qquad \ln|y| = \frac{1}{2}\ln(x^2+1) + C = \ln\sqrt{x^2+1} + C,
\end{aligned}$$

where C is any constant. Exponentiating both sides, we obtain

$$|y| = \mathrm{e}^{\ln\sqrt{x^2+1}+C} = \mathrm{e}^{\ln\sqrt{x^2+1}} \cdot e^C = A\sqrt{x^2+1} \qquad \Longleftrightarrow \qquad y = \pm A\sqrt{x^2+1}; \tag{2}$$

since C is any constant, $A = e^C$ is any *positive* constant. However, we divided both sides of (1) by y. This is 0 if $y = 0$ for all x; this gives rise to the only constant solution of the differential equation, which in turn corresponds to $A = 0$ in (2). So the general solution (i.e. the set of all solutions) of (1) is $\boxed{y(x) = C\sqrt{x^2+1}}$ where C is now any constant.

Note: It is good to check that the function $y = y(x)$ is indeed a solution of (1) by computing y', $(x^2+1)y'$, and xy and comparing the last two.

Section 7.3, Problem 30 (weabassign)

Find the orthogonal trajectories to the family of curves $y^2 = kx^3$. Make a large-size detailed sketch that includes at least 3 members of the original family of curves and at least 3 members of the orthogonal family of curves.

Differentiate the equation $y^2 = kx^3$ with respect to x, using chain rule and remembering that k is a constant:

$$2yy' = k \cdot 3x^2 = 3\frac{y^2}{x}\,, \tag{3}$$

since $k = y^2/x^3$. So our curves have slope $y' = (3/2)y/x$ at (x, y). The slopes of the orthogonal curves are the negative inverses of this; so they satisfy

$$y'_{\text{new}} = -\frac{2}{3} \cdot \frac{x}{y_{\text{new}}}. \tag{4}$$

Note that while y in (3) refers to the original curves, y_{new} in (4) refers to the orthogonal curves. It is the latter equation we need to solve to find the orthogonal curves (the general solution of the former, $y^2 = kx^3$, is already given).

Equation (4) is separable, so after writing $y'_{\text{new}} = \mathrm{d}y_{\text{new}}/\mathrm{d}x$, we can move everything involving y_{new} to LHS and everything involving x to RHS and then integrate:

$$\frac{\mathrm{d}y_{\text{new}}}{\mathrm{d}x} = -\frac{2}{3} \cdot \frac{x}{y_{\text{new}}} \iff y_{\text{new}}\,\mathrm{d}y_{\text{new}} = -\frac{2}{3}x\,\mathrm{d}x \iff \int y_{\text{new}}\,\mathrm{d}y_{\text{new}} = -\int \frac{2}{3}x\,\mathrm{d}x$$
$$\iff \frac{1}{2}y_{\text{new}}^2 = -\frac{1}{3}x^2 + C \iff 2x^2 + 3y_{\text{new}}^2 = A,$$

where $A = 6C$ is any constant. If $A < 0$, the last equation has no (real) solutions and so does not correspond to a curve in the real xy-plane. If $A = 0$, the "curve" is just the point $(0, 0)$. If $A > 0$, the corresponding curve in the xy-plane is the ellipse centered at $(0, 0)$, symmetric about the coordinate axes, and with the longer "horizontal radius" equal to $\sqrt{3/2}$ times the short "vertical radius"; see Figure 1. The original curves are the x-axis (for $k = 0$) and the graphs of the functions $y = C|x|^{3/2}$ with $C = \pm\sqrt{|k|} \neq 0$.

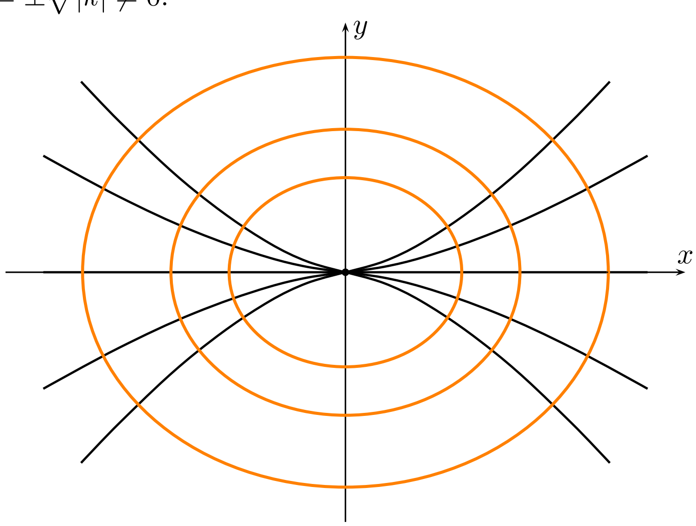

Figure 1: Graph for Section 7.3, Problem 30

Section 7.3, Problem 45 (webassign)

A tank contains 1000 L of brine with 15 kg of dissolved salt. Pure water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate.

(a) *How much salt is in the tank after t minutes?*
Let $y(t)$ be the amount of salt in the tank, in kgs, at time t, in minutes. Thus, $y(0)=15$ and

$$y'(t) = y'_{\text{in}}(t) - y'_{\text{out}}(t),$$

where

$$\begin{aligned} y'_{\text{in}}(t) &= (\text{flow rate of salt})_{\text{in}} = (\text{flow rate of solution})_{\text{in}} \cdot (\text{flow concentration})_{\text{in}} = 10 \cdot 0\% = 0; \\ y'_{\text{out}}(t) &= (\text{flow rate of salt})_{\text{out}} = (\text{flow rate of solution})_{\text{out}} \cdot (\text{flow concentration})_{\text{out}} . \end{aligned}$$

Since the solution in the tank is kept thoroughly mixed, the outgoing flow concentration is the same as the salt concentration in the tank:

$$(\text{flow concentration})_{\text{out}} = \frac{\text{amount of salt in tank}}{\text{volume in tank}} = \frac{y(t)}{1000},$$

since the volume of solution in the tank is kept constant at 1000 gallons. So,

$$y'_{\text{out}}(t) = 10 \cdot \frac{y(t)}{1000} = \frac{y(t)}{100}.$$

It follows that $y(t)$ is the solution to the initial-value problem

$$y'(t) = 0 - \frac{y(t)}{100} = -\frac{1}{100}y(t), \qquad y(0) = 15.$$

Since this is just the exponential decay equation, the solution to this initial-value problem is

$$y(t) = y(0)\mathrm{e}^{-\frac{1}{100}t} = \boxed{15\mathrm{e}^{-t/100}}$$

Alternatively, we can find the general solution of the differential equation and the particular solution satisfying the initial condition. Since the differential equation is separable, writing $y' = \mathrm{d}y/\mathrm{d}t$, moving everything involving y to LHS and everything involving t to the RHS, and integrating, we obtain

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{1}{100}y \quad &\Longleftrightarrow \quad \frac{\mathrm{d}y}{y} = -\frac{1}{100}\mathrm{d}t \quad \Longleftrightarrow \quad \int \frac{\mathrm{d}y}{y} = -\int \frac{1}{100}\mathrm{d}t \quad \Longleftrightarrow \quad \ln|y| = -\frac{t}{100} + C \\ &\Longleftrightarrow \quad \mathrm{e}^{\ln|y|} = \mathrm{e}^{-t/100+C} = \mathrm{e}^{-t/100} \cdot \mathrm{e}^{C} \quad \Longleftrightarrow \quad \ln|y| = \mathrm{e}^{C}\mathrm{e}^{-t/100} \\ &\Longleftrightarrow \quad y = A\mathrm{e}^{-t/100}, \end{aligned}$$

where C is any constant and $A = \pm\mathrm{e}^{-t/100}$ is any nonzero constant. However, we divided the equation by y, which is 0 if $y = 0$; this gives us the constant solution $y = 0$ of the differential equation, which corresponds to $A=0$. In order to find A, plug in the initial condition $(t, y)=(0, 15)$ into the last equation above:

$$15 = A\mathrm{e}^{-0/100} \quad \Longrightarrow \quad A = 15 \quad \Longrightarrow \quad \boxed{y(t) = 15\mathrm{e}^{-t/100}}$$

We could also find the particular solution by plugging in the initial condition $(t,y)=(0,15)$ into the first equation equation above that contains C:

$$\begin{aligned}\ln|15| = -\frac{0}{100}+C \quad &\Longrightarrow \quad C=\ln 15 \quad \Longrightarrow \quad \ln|y| = -\frac{t}{100}+\ln 15\\ &\Longrightarrow \quad \mathrm{e}^{\ln|y|} = \mathrm{e}^{-\frac{t}{100}+\ln 15} = \mathrm{e}^{-\frac{t}{100}}\cdot \mathrm{e}^{\ln 15} = 15\mathrm{e}^{-\frac{t}{100}}\\ &\Longrightarrow \quad |y| = 15\mathrm{e}^{-\frac{t}{100}} \quad \Longrightarrow \quad y=\pm 15\mathrm{e}^{-\frac{t}{100}}.\end{aligned}$$

Since $y(t)$ cannot be negative, the sign above must be $+$ and we recover the same formula for $y(t)$.

Note: As a reality check, note that $y(t)$ approaches 0 as $t\longrightarrow\infty$, as expected because the salt gets washed out with the pure water being poured into the tank.

(b) *How much salt is in the tank after 20 minutes?*
Since in the above formulas for $y(t)$ the time t is measured in minutes, we can simply plug in $t=20$:

$$y(20) = 15\mathrm{e}^{-20/100} = \boxed{15\mathrm{e}^{-1/5} \approx 12.28 \text{ kg}}$$

Note the units.