MAT 127: Calculus C, Spring 2015
Solutions to a HW5 Problem

Below you will find a detailed solution to a problem from HW5. Since it was part of the WebAssign portion of this homework, your version of this problem likely had different numerical coefficients. However, the principles behind the solutions and their structure are as described below.

Section 7.6, Problem 10 (webassign)

Populations of aphids A and ladybugs L are modeled by the following equations:

$$\begin{cases} \frac{\mathrm{d}A}{\mathrm{d}t} = 2A - .01AL \\ \frac{\mathrm{d}L}{\mathrm{d}t} = -.5L + .0001AL \end{cases} \tag{1}$$

(a) *Find all equilibrium solutions and explain their significance.*

The equilibrium (constant) solutions of (1) are pairs of numbers (A, L) such that

$$\begin{cases} 2A - .01AL = \frac{A}{100}(200 - L) = 0 \\ -.5L + .0001AL = -\frac{L}{10{,}000}(5000 - A) = 0 \end{cases}$$

These pairs satisfy both of the following conditions

$$\begin{cases} A = 0 \text{ or } 200 - L = 0 \\ L = 0 \text{ or } 5000 - A = 0. \end{cases}$$

If we choose the first option on the first line, i.e. $A=0$, then we must choose the first option on the second line, i.e. $L=0$ (because the second option on the second line contradicts our choice from the first line). This gives the equilibrium solution $(A, L)=(0, 0)$, which means there are no aphids or ladybugs ever. On the other hand, if we choose the second option from the first line, i.e. $L=200$, then we must choose the second option on the second line as well, i.e. $A=5000$. So the only other equilibrium solution is $(A, L)=(5000, 200)$; so 5000 aphids are precisely enough to support 200 ladybugs and be contained by them.

Note: A more systematic approach to extracting the equilibrium solutions from the last system of equations above is to write a system of equation for *each* pair consisting of a condition from the first line and a condition from the second line. In this case, we get $2\cdot 2=4$ systems:

$$\begin{cases} A = 0 \\ L = 0 \end{cases} \qquad \begin{cases} A = 0 \\ 5000 - A = 0 \end{cases} \qquad \begin{cases} 200 - L = 0 \\ L = 0 \end{cases} \qquad \begin{cases} 200 - L = 0 \\ 5000 - A = 0 \end{cases}$$

We must then find ALL solutions (A, L) of *each* of these systems. In this case, the second and third systems of equations have no solutions, while the first and the forth give us $(A, L) = (0, 0)$ and $(A, L) = (5000, 200)$, respectively.

(b) *Find an expression for* $\mathrm{d}L/\mathrm{d}A$.

Just divide the second equation in (1) by the first:

$$\frac{\mathrm{d}L}{\mathrm{d}A} = -\frac{L}{100A} \cdot \frac{5000 - A}{200 - L}.$$

(c) *The figure on p546 in the book shows the direction field for the differential equation in part (b). Use it to sketch a phase plane portrait. What do the phase trajectories have in common?*

The trajectories for the system of the differential equations in (1) travel along the solution curves for the differential equation in (b). These solution curves are everywhere tangent to the little slope lines. In this case, the solution curves are loops going around the equilibrium point $(A, L) = (5000, 200)$, as can be seen from the direction field and is proved in Problem E below (the non-trivial part is that these curves are necessarily closed, i.e. circle back to themselves). If $A = 5000$ and $L \in (0, 200)$, i.e. at a point directly below this equilibrium point, $\mathrm{d}A/\mathrm{d}t > 0$ by the first equation in (1), while $\mathrm{d}L/\mathrm{d}t = 0$. Thus, at t increases, the point $(A(t), L(t))$ travels *counter-clockwise* along such a closed curve.

(d) *Suppose that at time* $t = 0$ *there are 1000 aphids and 200 ladybugs. Draw the corresponding phase trajectory and use it do describe how both population change.*

This trajectory starts at $(A, L) = (1000, 200)$; this point lies 1/5 of the way from the y-axis to the equilibrium point $(5000, 200)$. By part (c), this trajectory then circles around the point $(5000, 200)$ counter-clockwise. So at first A increases, while L decreases. The trajectory reaches its lowest point when $A = 5000$ (at which point L looks like it might around 100); A then continues to increase, while L starts to increase as well. The trajectory reaches its right-most point when $L = 200$, while A looks like it might be around 15000; A then starts to decrease, while L continues to increases. The trajectory reaches its highest point when $A = 5000$ (at which point L looks like it might around 300); A then continues to decrease, while L starts to decrease as well. The trajectory reaches its left-most point when it returns to the starting point $(A, L) = (1000, 200)$, after which the entire cycle repeats. This is illustrated in the first diagram in Figure 1.

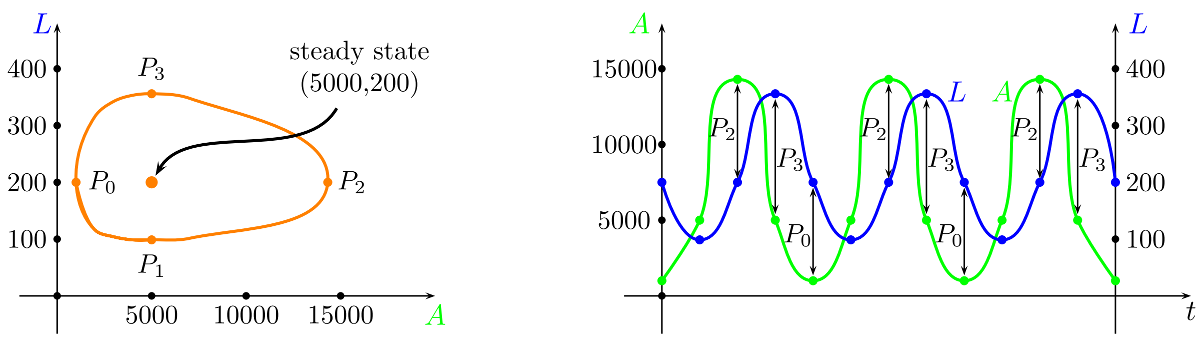

Figure 1: The left diagram shows a phase trajectory. The right diagram shows the corresponding graphs of the functions $A(t)$ and $L(t)$.

Note: we can find an equation for the curve traced by the above trajectory by solving the separable differential in (b) and using the initial condition $(A, L)=(1000, 200)$ to determine the constant. In fact, we can simply use the solution Problem E below, with

$$a = 2,\quad b = \frac{1}{100},\quad c = \frac{1}{2},\quad d = \frac{1}{10,000} \quad \Longrightarrow \quad L^2 e^{-L/100} = CA^{-1/2} e^{A/10000}$$

$$\Longrightarrow \quad 200^2 e^{-200/100} = C \cdot 1000^{-1/2} e^{1000/10000}$$

$$\Longrightarrow \quad C = 4\sqrt{10} \cdot 10^5 \cdot e^{-21/10}$$

$$\Longrightarrow \quad L^2 e^{-L/100} = 4 \cdot 10^{11/2} e^{-21/10} A^{-1/2} e^{A/10000}.$$

From this, we find that the largest possible value of A is roughly 14302, while the minimum and maximum values of L are roughly 98 and 356, respectively.

(e) *Use part (d) to make rough sketches of the aphid and ladybug populations as functions of t. How are the graphs related to each other?*

First, mark the key points on the phase trajectory in the order they are traversed as t increases (counter-clockwise in the first diagram in Figure 1). These are

- the left-most point $P_0 = (1000, 200)$;
- the lowest point $P_1 \approx (5000, 100)$;
- the right-most point $P_2 \approx (15000, 200)$;
- the highest point $P_3 \approx (5000, 350)$.

Note that both coordinates of P_0 are exact, since this initial point is specified. The first coordinates of P_1 and P_3 are also exact and can be determined from the second equation in (1), since this is where $\mathrm{d}L/\mathrm{d}t = 0$. The second coordinate of P_2 is exact as well and can be determined from the first equation in (1), since this is where $\mathrm{d}A/\mathrm{d}t = 0$. The graphs of $A = A(t)$ and $L = L(t)$ can now be sketched by marking the coordinates of each of the key points of the trajectory on a diagram with horizontal t-axis and two separate vertical axes: A-axis and L-axis. The first coordinates then should be connected by one curve, corresponding to the graph of $A(t)$, while the second coordinates should be connected by another curve, corresponding to the graph of $L(t)$. The two graphs should have no other maxima or minima. While both graphs start at $t=0$, the intermediate t-values cannot be determined from the phase trajectory and so should not be marked on the t-axis. What matters is that the values of A and L for the marked points lie on the same vertical lines; they correspond to the same moments in time, but what these "moments in time" are cannot be determined (except for $t=0$). However, after the A and R return to their starting values, the cycle repeats exactly, taking the same amount of time from the P_0-coordinates to the P_1-coordinates as the first time, and so on.

A rough way in which the two graphs are related is that the L-graph (blue) is a "quarter" of a cycle behind the A-graph (green): the maxima and minima of the former occur a bit after the maxima and minima of the latter.

Note: In order to avoid mixing up the first coordinates (that are used for the A-graph) and the second coordinates (that are used for the L-graphs), either mark them in different colors or with dots and stars, etc. Do not forget to label the axes (t, A, and L in this case) and marked the appropriate scales on the vertical (A and L) axes; these axes should have the same points marked as the corresponding axes in the first diagram in Figure 1. However, the t-axis should carry **no** scale markings (e.g. $t = 1$), since the values of t at which the maxima and minima of $A(t)$ and $R(t)$ occur in the second diagram in Figure 1 cannot be determined from the phase trajectory in the first diagram in Figure 1.