REVIEW FOR MIDTERM I: MAT 310

(1) Let V denote a vector space over the field F; let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ denote vectors in V; let $\mathbf{a}$,$\mathbf{b}$ denote the two row vectors (a_1, a_2, a_3), (b_1, b_2, b_3) in F^3.

(a) Complete the definition: *S is a linearly independent set if .....*
Solution: *whenever $\sum_{i=1}^{3} x_i \mathbf{v}_i = \mathbf{0}$, for $x_i \in F$, then $x_i = 0$ for all i.*

In parts (b)(c) below assume that S is a linearly independent set.

(b) Show (directly from the definition of *linearly independent*) that the two vectors $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 + 3\mathbf{v}_3$ and $\mathbf{w} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - \mathbf{v}_3$ are linearly independent.
Solution: For any $x, y \in F$ we have $x\mathbf{v} + y\mathbf{w} = (x + 2y)\mathbf{v}_1 + (-x + 3y)\mathbf{v}_2 + (3x - y)\mathbf{v}_3$. Thus, if $x\mathbf{v} + y\mathbf{w} = \mathbf{0}$ then

$$x + 2y = 0, -x + 3y = 0, 3x - y = 0$$

(since S is assumed to be linearly independent). Solving these last three equations yields $x = 0 = y$.

(c) Show that the two vectors $\mathbf{v} = \sum_{i=1}^{3} a_i \mathbf{v_i}$ and $\mathbf{w} = \sum_{i=1}^{3} b_i \mathbf{v}_i$ are linearly independent in V iff the two vectors $\mathbf{a}, \mathbf{b}$ are linearly independent in F^3.
Soltution: If $\mathbf{v}$ and $\mathbf{w}$ are dependent then one is a scalar multiple of the other; e.g. $\mathbf{v} = x\mathbf{w}$ for $x \in F$. Thus $\mathbf{0} = \mathbf{v} - x\mathbf{w} = \sum_{i=1}^{3}(a_i - xb_i)\mathbf{v}_i$; which implies that $a_i - xb_i = 0$ for all i (since S is an independent set). Thus $a_i = xb_i$ for all i; so $\mathbf{a} = x\mathbf{b}$, showing that $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent.

Similarly, if $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent, you can reverse the argument just given to show that $\mathbf{v}$ and $\mathbf{w}$ are dependent.

(2) Let X, Y denote vector subspaces of $\mathbb{R}^4$.

(a) Set $S = X \cup Y$. Show that $X + Y = V$ iff span(S)=V.
Solution: $X+Y = V$ means that every vector $\mathbf{v} \in V$ can be written as $\mathbf{v} = \mathbf{x} + \mathbf{y}$ where $\mathbf{x} \in X$ and $\mathbf{y} \in Y$. Note that $\mathbf{x}, \mathbf{y} \in S$; thus $\mathbf{x}+\mathbf{y} \in span(S)$. So every vector $\mathbf{v} \in V$ is in span(S). Note that the span of any subset of V is a vector subspace of V; thus span(S) $\subset V$.

It remains to show that if span(S)=V then $X + Y = V$.

(b) Suppose that $\mathbf{x}_1 = (2, 6, 2, 10), \mathbf{x_2} = (2, 0, 0, 1)$ is a generating set for X and $\mathbf{y}_1 = (0, 5, 4, 8)$,$\mathbf{y}_2 = (0, 1, -2, 1)$ is generating set for Y.

(i) Explain why the subset $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2\}$ is a generating set for $X + Y$.

Solution: If $\mathbf{v} \in X+Y$ then $\mathbf{v} = \mathbf{x}+\mathbf{y}$, where $\mathbf{x} \in X$ and $\mathbf{y} \in Y$. But $\mathbf{x} = \sum_{i=1}^{2} a_i\mathbf{x}_i$ and $\mathbf{y} = \sum_{i=1}^{2} b_i\mathbf{y}_i$. Thus $\mathbf{v} = \sum_{i=1}^{2} a_i\mathbf{x}_i + \sum_{i=1}^{2} b_i\mathbf{y}_i$, showing that $\mathbf{v}$ is a linear combination of vectors in S.

(ii) Find a subset $\alpha \subset S$ which is a basis for $X+Y$; and compute $dim(X+Y)$.

Solution: $\alpha = \{\mathbf{x}_2, \mathbf{y}_1, \mathbf{y_2}\}$; $dim(X+Y) = 3$.

(iii) Extend the basis α for $X+Y$ to a basis β for all of $\mathbb{R}^4$.

Solution: Set $\beta = \alpha \cup \{(1,0,0,0)\}$.

(c) Let α and β denote basis for X and Y respectively and set $\gamma = \alpha \cup \beta$. Show that V is the direct sum of X and Y iff γ is a basis for V.

Solution: I will do half of this.

Set $\alpha = \{\mathbf{x}_1, ..., \mathbf{x}_m\}$ and $\beta = \{\mathbf{y}_1, ..., \mathbf{y}_n\}$, where $m+n=4$. We will assume that γ is a basis for V and show that V is the direct sum of X and Y, i.e. we will show that $V = X+Y$ and that $X \cap Y = \{\mathbf{0}\}$.

Since γ is a basis for V we have that span$(\gamma) = V$. Thus any vector $\mathbf{v} \in V$ can be written as $\mathbf{v} = \sum_{i=1}^{m} a_i\mathbf{x}_i + \sum_{j=1}^{n} b_j\mathbf{y}_j$; so $\mathbf{v} = \mathbf{x}+\mathbf{y}$ where $\mathbf{x} = \sum_{i=1}^{m} a_i\mathbf{x}_i$ is in X and $\mathbf{y} = \sum_{j=1}^{n} b_j\mathbf{y}_j$ is in Y. This shows that $\mathbf{v} \in X+Y$; thus $V = X+Y$.

If $\mathbf{v} \in X \cap Y$ then $\mathbf{v} = \sum_{i=1}^{m} a_i\mathbf{x}_i$ and $\mathbf{v} = \sum_{j=1}^{n} b_j\mathbf{y}_j$; thus $\mathbf{0} = \mathbf{v} - \mathbf{v} = \sum_{i=1}^{m} a_i\mathbf{x}_i + \sum_{j=1}^{n} -b_j\mathbf{y}_j$, from which we conclude that $a_i = 0$ and $-b_j = 0$ for all i,j (because γ is basis for V and is therefore a linearly independent set). This shows that $\mathbf{v} = \mathbf{0}$; thus $X \cap Y = \mathbf{0}$.

(3) Let $S = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ denote a finite subset of the vector space V.

(a) Suppose S is not an independent set. Then prove (from basics) that there is a subset $S' \subset S$ with $S' \neq S$, such that $span(S') = span(S)$.

Solution: There is a non-trivial linear combination $\sum_{i=1}^{n} a_i\mathbf{v}_i$ which equals $\mathbf{0}$; i.e. one of the coefficients $a_j \neq 0$. Thus we can solve the equation

$$\sum_{i=1}^{n} a_i\mathbf{v}_i = \mathbf{0}$$

for $\mathbf{v}_j$ to get that

$$\mathbf{v}_j = \sum_{i\neq j} (-a_j)^{-1} a_i\mathbf{v}_i. \tag{1}$$

Set $S' = \{\mathbf{v}_1, ..., \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, ..., \mathbf{v}_n\}$.

To see that span(S')=span(S), let $\mathbf{v} \in span(S)$. Then

$$\mathbf{v} = \sum_{i=1}^{n} x_i\mathbf{v}_i. \tag{2}$$

Combining equations (1) and (2) we get that

$$\mathbf{v} = \sum_{i\neq j} (x_i + x_j(-a_j)^{-1}a_i)\mathbf{v}_i,$$

which shows that $\mathbf{v} \in span(S')$.

(b) Suppose S is an independent set such that $span(S) \neq V$. Then prove (from basics) that there is a vector v_{n+1} which is in V but is not in S, such that $S \cup \{\mathbf{v}_{n+1}\}$ is an independent set.
Solution: Choose any vector $\mathbf{v}_{n+1} \in V$ which is not in span(S). If $S \cup \{\mathbf{v}_{n+1}\}$ is linearly dependent then we have $\sum_{i=1}^{n+1} a_i v_i = \mathbf{0}$, where $a_j \neq 0$ for some j. If j=n+1, then we can solve this last equation for $v_{n+1} = \sum_{i=1}^{n}(-a_{n+1})^{-1} a_i \mathbf{v}_i$, showing that $v_{n+1} \in span(S)$ — a contradiction. If $j \neq n+1$, then $a_{n+1} = 0$ and $\sum_{i=1}^{n} a_i \mathbf{v}_i = \mathbf{0}$ with $a_i \neq 0$ for some $1 \leq i \leq n$, showing that S is a linearly dependent set — which is again a contradiction.

(4) Consider the subset $S = \{2x^3 + x^2, 2x^3 - 1, x^2 - x, x + 1\}$ of the vector space $P_4(\mathbb{R})$.

(a) Find a basis α for span(S); compute $dim(span(S))$.
Solution: $\alpha = \{2x^3 - 1, x^2 - x, x + 1\}$. $dim(span(S)) = 3$.

(b) Extend α to a basis β for all of $P_4(\mathbb{R})$.
Solution: $\beta = S \cup \{x^3\}$.

(5) Let V, W denote real vector spaces.

(a) Complete the following definition: *A function $T : V \longrightarrow W$ is a linear transformation if .....*

(b) Argue directly from the definition in part (a), prove that if $T : V \longrightarrow W$ is a linear transformation then $T(\sum_{i=1}^{3} a_i \mathbf{v}_i) = \sum_{i=1}^{3} a_i T(\mathbf{v}_i)$ is true for any real numbers a_i and any vectors $\mathbf{v}_i \in V$.

(c) If $T : V \longrightarrow W$ is a linear transformation, then give the definition for $N(T)$ — the null space of T. Prove (from basics) that N(T) is a subspace of V.

(6) Let $T : \mathbb{R}^6 \longrightarrow \mathbb{R}^3$ denote a linear transformation such that $T((1,0,0,0,0,0)) = (3,-1,0)$, $T((1,1,1,1,1,1)) = (-2,1,3)$, $T((0,0,1,1,1,1) = (0,1,1)$. Compute $dim(N(T))$.
Solution: The vectors $(3,-1,0), (-2,1,3), (0,1,1)$ are all in $R(T)$. These vectors are independent vectors in $\mathbb{R}^3$; thus $dim(R(T)) \geq 3$. But $dim(R(T)) \leq dim(\mathbb{R}^3) = 3$. So $dim(R(T)) = 3$. Finally $dim(\mathbb{R}^6) = dim(N(T)) + dim(R(T))$, i.e. $6 = dim(N(T)) + 3$. Thus $dim(N(T)) = 3$.

(7) Is there a linear transformation $T : (Z_2)^5 \longrightarrow P_3(Z_2)$ which satisfies $T((1,1,0,0,1)) = x^3 + x$, $T((0,0,0,1,1)) = x^2 + 1$, $T((1,1,0,1,0)) = x^3 + 1$? (If yes, then describe it; if no then prove that there is no such linear map.)
Solution: The answer is No.

Note that $(1,1,0,0,1)+(0,0,0,1,1)=(1,1,0,1,0)$. Thus, if we assume that T is linear, then we have

$$T((1,1,0,1,0)=T((1,1,0,0,1)+(0,0,0,1,1))= \tag{1}$$

$$T((1,1,0,0,1))+T((0,0,0,1,1))=(x^3+x)+(x^2+1)=x^3+x^2+x+1.$$

But we also are given that

$$T((1,1,0,1,0))=x^3+1. \tag{2}$$

Equations (1)(2) contradict one another.

(8) Let V denote the vector space of all continuous real valued functions defined on the real line. Are the functions $t^2e^t, e^t, 2^t$ independent vectors in V?

Solution: Yes, they are idependent. If

$$at^2e^t+be^t+c2^t=0 \tag{1}$$

then by applying the differential operator $(D-D^0)^3$ to equation (1) we get

$$(ln(2)-1)^3c2^t=0, \tag{2}$$

which implies that

$$c=0. \tag{3}$$

By applying the differential operator $D-D^0$ to equation (1), and using equation (3), we get that

$$2ate^t=0, \tag{4}$$

which implies that

$$a=0. \tag{5}$$

Now combining (1)(3)(5) we also get that $b=0$.

(9) Let C^∞ denote the vector space (over the complex numbers) of infinitely differentiable complex valued functions defined on the real line. Find the general solution to the homogeneous differential equation

$$p(D)x=0$$

where

$$p(t)=(t^2+1)^2(t^2+4t+3).$$

Solution: The polynomial $p(t)$ factors completely as $p(t)=(t+i)^2(t-i)^2(t+1)(t+3)$. Thus a basis for the solution space of $p(D)X=0$ is $\{e^{-it}, te^{-it}, e^{it}, te^{it}, e^{-t}, e^{-3t}\}$; the general solution has the form

$$x(t)=a_1e^{-it}+a_2te^{-it}+b_1e^{it}+b_2te^{it}+ce^{-t}+de^{-3t}.$$

(10) Set $\alpha=\{1,x,x^2\}$ and $\beta=\{x^2+x, x-1, x^2+x+1\}$. Note that both α and β are basis for $P_2(\mathbb{R})$. Let γ denote a third basis for $P_2(\mathbb{R})$ such that the 3×3 matrix

$$\begin{matrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{matrix}$$

is the "change of coordinate matrix" that changes the γ-coordinates of a vector to the β-coordinates.

(a) Compute $[x]_\beta$. **Solution:** $[x]_\beta$ is the column vector

$$\begin{matrix} -1 \\ 1 \\ 1 \end{matrix}$$

(b) What are the vectors in γ?
Solution: $\gamma = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where

$$\mathbf{v}_1 = (x^2 + x) + 2(x - 1)$$

$$\mathbf{v}_2 = (x^2 + x) + (x^2 + x + 1)$$

$$\mathbf{v}_3 = 2(x^2 + x) + (x - 1) + (x^2 + x + 1)$$

(c) Compute the change of coordinate matrix that changes γ-coordinates to α-coordinates.
Solution: This is the matrix $[id]_\gamma^\alpha$. Note that

$$[id]_\gamma^\alpha = [id]_\beta^\alpha [id]_\gamma^\beta \qquad (1)$$

and $[id]_\beta^\alpha$ is the 3×3-matrix

$$\begin{matrix} 0 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}$$

Now compute $[id]_\gamma^\alpha$ by multiplying the above two matrices.

(11) Consider the linear transformation $T : P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ defined by $T(p(x)) = 3p''(x) - 2p'(x) + p(x)$.

(a) Is T invertible (why or why not)?
Solution: $T(1) = 1, T(x) = x - 2, T(x^2) = x^2 - 4x + 6$. Thus $S = \{1, x-2, x^2-4x+6\}$ is a subset $R(T)$. Note that S is an independent set, so $dim(R(T)) \geq 3$. But $dim(R(T)) \leq dim(P_2(\mathbb{R})) = 3$; so $dim(R(T)) = 3 = dim(P_2(\mathbb{R}))$ and $R(T) = P_2(\mathbb{R})$. So $T : P_2(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ is an onto linear transformation, which implies that it is invertible (why?).

(b) Compute $[T]_\alpha$ and $[T]^\alpha_\beta$, where α and β are as in problem (10) above.
Solution: The computations for $T(1), T(x), T(x^2)$ givn in part (a) above, show that $[T]_\alpha$ is equal to the matrix

$$\begin{matrix} 1 & -2 & 6 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{matrix}$$

To get $[T]^\alpha_\beta$, note that

$$[T]^\alpha_\beta = [T]_\alpha [id]^\alpha_\beta.$$

So to get the $[T]^\alpha_\beta$ just multiply the proceeding matrix with the matrix $[id]^\alpha_\beta$ which was computed in (10)(c) above.

(12) Determine whether each of the following statements is true or false.

(a) If a linear transformation $T : \mathbb{R}^3 \longrightarrow P_3(\mathbb{R})$ is one-one then it is an isomorphism.
(b) Every matrix $A \in M_{5\times 5}(\mathbb{C})$ is the product of a finite number of elementary matrices.($\mathbb{C}$=complex numbers.)
(c) If dim(V)=dim(W) for two vectors spaces V,W over the same field F, then V must be isomorphic to W.
(d) For any $A, B \in M_{2\times 2}(F)$ we must have AB=BA. (F=field.)
(e) For any $A \in M_{n\times n}(F)$ if A is invertible then A^k is also invertible for each positive integer k. (F=field.)
(f) Let $T : V \longrightarrow V$ denote any linear operator on the finite dimensional vectors space V. Then $rank(T) \leq rank(T^2)$.
(g) $A, B \in M_{n\times n}(\mathbb{R})$ are similar matrices iff rank(A)=rank(B).
(h) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is an independent set in the vector space V then dim(V)≤ 4.
(i) The matrix $A \in M_{n\times n}(F)$ is invertible iff rank(A)=n.

(13) Let A denote the following 3×3 matrix over the real numbers:

$$\begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{matrix}$$

(a) Write A as a finite product of elementary matrices.
(b) Compute A^{-1}.

(14)

(a) Let A denote the 4×4 matrix with real number entries

$$\begin{matrix} 1 & 2 & 2 & 1 \\ -1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{matrix}$$

and let B denote the 4×4 matrix with real number entries

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 6 & 0 & 2 & 3 \\ 3 & -1 & 0 & 2 \end{matrix}$$

Is A similar to B over the real numbers? (**Hint:** What are the ranks of A and B?)

(b) Is the the 3×3 matrix

$$\begin{matrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{matrix}$$

similar (over the real numbers) to an elementary matrix?

(c) Suppose that the matrices $A, B \in M_{n \times n}(F)$ are similar over the field F. Then show that the matrices A^2, B^2 are also similar over F.