1. Vector Bundles

Convention: All manifolds here are Hausdorff and paracompact. To make our life easier, we will assume that all topological spaces are homeomorphic to CW complexes unless stated otherwise.

The definition of a smooth vector bundle in some sense is similar to the definition of a smooth manifold except that 'chart' is now replaced with 'trivialization'.

Definition 1.1. A **smooth real vector bundle of rank** k **over a base** B is a smooth surjection $\pi : V \longrightarrow B$ between smooth manifolds, an open cover $(U_i)_{i \in S}$ of M, and homeomorphisms $\tau_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^k$ called **trivializations** satisfying the following properties:

(1) Let $\pi_{U_i} : U_i \times \mathbb{R}^k \longrightarrow U_i$ be the natural projection. Then $\pi|_{\pi^{-1}(U_i)} = \pi_i \circ \tau_i$. In other words, we have the following commutative diagram:

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\tau_i} & U_i \times \mathbb{R}^k \\ & {\scriptstyle \pi|_{\pi^{-1}(U_i)}} \searrow \quad \swarrow {\scriptstyle \pi_{U_i}} & \\ & U_i & \end{array}$$

(2) The **transition maps**

$$\tau_i \circ \tau_j^{-1} : (U_i \times U_j) \times \mathbb{R}^k \longrightarrow (U_i \times U_j) \times \mathbb{R}^k$$

are smooth maps satisfying:

$$\tau_i \circ \tau_j^{-1}(x, z) = (x, \Phi_{ij}(x).z)$$

where

$$\Phi_{ij} : U_i \cap U_j \longrightarrow GL(\mathbb{R}^k)$$

is a smooth map.

We will call the maps $\Phi_{ij} : U_i \cap U_j \longrightarrow GL(\mathbb{R}^k)$ **transition data**.

Remark 1 Instead of writing $\pi : V \longrightarrow B$, $(U_i)_{i \in S}$, $(\tau_i)_{i \in S}$ we will just write $\pi : V \longrightarrow B$ for a vector bundle.

Remark 2: Note that we can change the group GL to other groups such as $SL(\mathbb{R}^k)$ (matrices of determinant 1) or $O(\mathbb{R}^k)$, orthogonal matrices or $GL(n, \mathbb{C})$ complex $n \times n$ matrices where we identify $\mathbb{R}^k \cong \mathbb{C}^{k/2}$ (if k is even).

Remark 3: If we *formally* replace $\mathbb{R}^k$ with a smooth manifold F and the group $GL(n, \mathbb{R})$ with a group G and a group homomorphism $G \to Diff(F)$ then we get the definition of a **fiber bundle with structure group** G.

Remark 4: Note that the trivializations $(\tau_i)_{i \in S}$ form an atlas on V and hence uniquely specify the smooth structure on E. Hence in the above definition, we only need to specify that V is a set and the formally replace the words 'smooth surjection' with 'surjection'.

Remark 5: We can also have that E,B are topological spaces and all the maps are continuous including the transition maps. Then this is a **topological vector bundle.**

Exercise: give a definition of a topological vector bundle.

Technically, a smooth real vector bundle of rank k has an *equivalence class* of open covers and trivializations (just as a manifold really consists of a set with an equivalence class of charts). Two such sets of open covers $(U_i)_{i \in S}$, $(U'_i)_{i \in S'}$ and trivializations

$$(\tau_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^k)_{i \in S}, \quad (\tau'_i : \pi^{-1}(U'_i) \longrightarrow U'_i \times \mathbb{R}^k)_{i \in S'}$$

associated to these open covers are equivalent if their union satisfies (1) and (2). In other words,

$$\tau'_i \circ \tau_j^{-1} : (U'_i \times U_j) \times \mathbb{R}^k \longrightarrow (U'_i \times U_j) \times \mathbb{R}^k$$

are smooth maps satisfying:

$$\tau'_i \circ \tau_j^{-1}(x,z) = (x, \Phi_{ij}(x).z)$$

where

$$\Phi_{ij} : U'_i \cap U_j \longrightarrow GL(\mathbb{R}^k)$$

is a smooth map for all $i \in S$ and $j \in S'$.

Definition 1.2. A **trivialization of** $\pi : V \longrightarrow B$ **over an open set** $U \subset B$ of a smooth vector bundle as above is a smooth map:

$$\tau : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$$

satisfying

$$\tau_i \circ \tau^{-1} : (U_i \cap U) \times \mathbb{R}^k \longrightarrow (U_i \cap U) \times \mathbb{R}^k, \quad \tau_i \circ \tau^{-1}(x,z) = (x, \Phi_i(z))$$

for some smooth $\Phi_i : U_i \cap U \longrightarrow GL(\mathbb{R}^k)$.

In Milnor's book this is called a **local coordinate system**.

Example 1.3. The **trivial** $\mathbb{R}^k$ **bundle** over B is the smooth map $\pi_B : B \times \mathbb{R}^k \longrightarrow B$ where π_B is the natural projection map and the open cover is just B and the trivialization τ is just the identity map.

Example 1.4. The **tangent bundle** $\pi_{TB} : TB \longrightarrow B$ of a manifold B is constructed as follows:

Here TB is the set of equivalence classes of smooth maps $\gamma : \mathbb{R} \longrightarrow B$ where two such paths γ_1, γ_2 are **tangent** if $\gamma_1(0) = \gamma_2(0)$ and $\frac{d}{dt}(\psi \circ \gamma_1(t))|_{t=0} = \frac{d}{dt}(\psi \circ \gamma_2(t))|_{t=0}$ for some chart ψ of B containing $\gamma_1(0)$.

The open cover $(U_i)_{i \in S}$ of B consists of the domains of charts $\psi_i : U_i \longrightarrow \mathbb{R}^k$ on B. The trivialization $\tau_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^k$ is the map $\tau_i(\gamma) = (\gamma(0), \frac{d}{dt}(\psi \circ \gamma)|_{t=0})$.

Vector bundles can be built just from the data Φ_{ij} as in Definition 1.1. Note that this is a very similar procedure to constructing a manifold for a bunch of maps (corresponding to atlases) and transition functions 'gluing' these atlases together.

Constructing vector bundles from transition data: This is called the **Fiber bundle Construction Theorem** (in the case of fiber bundles).

Let B be a smooth manifold and $(U_i)_{i \in S}$ an open cover and let

$$\Phi_{ij} : U_i \cap U_j \longrightarrow GL(\mathbb{R}^k), \quad i,j \in S$$

be smooth maps satisfying the **cocycle condition:**

$$\Phi_{ij}(x)\Phi_{jk}(x) = \Phi_{ik}(x) \quad \forall x \in U_i \cap U_j \cap U_k.$$

Then we can construct a vector bundle with associated open cover $(U_i)_{i \in S}$ and transition data Φ_{ij} as follows: Here we define

$$V \equiv \left(\sqcup_{i \in S} U_i \times \mathbb{R}^k \right) / \sim$$

where $(u,x) \sim (u, \Phi_{ij}(x))$ for all $u \in U_i \cap U_j$, $x \in \mathbb{R}^k$ and all $i,j \in S$. Here $\pi : V \longrightarrow B$ sends $(u,x) \in U_i \times \mathbb{R}^k$ to $u \in B$.

Exercise: Show that V is a *Hausforff paracompact* C^∞ manifold with atlas given by $U_i \times F$ and then show that $\pi : V \longrightarrow B$ is a vector bundle.

Definition 1.5. A **homomorphism** between two vector bundles $\pi_1 : V_1 \longrightarrow B, \quad \pi_2 : V_2 \longrightarrow B$ of rank k over the base B is a smooth map $\Psi : V_1 \longrightarrow V_2$ so that

(1) The diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{\ \Psi\ } & V_2 \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & B & \end{array}$$

commutes and

(2) Any two trivializations

$$\tau_1 : \pi_1^{-1}(U_1) \longrightarrow U_1 \times \mathbb{R}^k, \quad \tau_2 : \pi_1^{-1}(U_2) \longrightarrow U_2 \times \mathbb{R}^k,$$

satisfy

$$\tau_2 \circ \Psi \circ \tau_1^{-1} : (U_1 \cap U_2) \times \mathbb{R}^k \longrightarrow (U_1 \cap U_2) \times \mathbb{R}^k, \quad \tau_i \circ \tau^{-1}(x,z) = (x, \Phi(z))$$

for some smooth $\Phi : U_1 \cap U_2 \longrightarrow Hom(\mathbb{R}^k)$.

A **isomorphism** is a homomorphism Ψ which is a diffeomorphism.
(Ex: Show that it's inverse is also a vector bundle homomorphism).

Example 1.6. Let $\pi_B : B \times \mathbb{R}^k \longrightarrow B$ be a trivial vector bundle 1.3 Then for any smooth map $\Phi : B \longrightarrow Hom(\mathbb{R}^k; \mathbb{R}^k)$, we have a vector bundle homomorphism

$$\Psi : B \times \mathbb{R}^k \longrightarrow B \times \mathbb{R}^k, \quad \Psi(x,z) = (x, \Phi(z)).$$

This is an isomorphism if $\mathrm{Im}(\Phi) \subset GL(\mathbb{R}^k)$.

Definition 1.7. A **vector subbundle** of a vector bundle $\pi_2 : V_2 \longrightarrow B$ is a submanifold $V_1 \subset V_2$ so that $\pi_2|_{V_1} : V_1 \longrightarrow B$ is a vector bundle and the inclusion map $V_1 \hookrightarrow V_2$ is a vector bundle homomorphism.

Definition 1.8. If $\phi : B_1 \longrightarrow B_2$ is a smooth map and $\pi_1 : V_1 \longrightarrow B_1, \quad \pi_2 : V_2 \longrightarrow B_1$ are vector bundles of rank k then a smooth map $\Psi : V_1 \longrightarrow V_2$ is a **bundle map covering** ϕ if

(1)
$$\begin{array}{ccc} V_1 & \xrightarrow{\ \Psi\ } & V_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ B_1 & \xrightarrow{\ \phi\ } & B_2 \end{array}$$

commutes and

(2) Any two trivializations

$$\tau_1 : \pi_1^{-1}(U_1) \longrightarrow U_1 \times \mathbb{R}^k, \quad \tau_2 : \pi_1^{-1}(U_2) \longrightarrow U_2 \times \mathbb{R}^k,$$

satisfy

$$\tau_2 \circ \Psi \circ \tau_1^{-1} : (U_1 \cap U_2) \times \mathbb{R}^k \longrightarrow (U_1 \cap U_2) \times \mathbb{R}^k, \quad \tau_2 \circ \tau_1^{-1}(x,z) = (x, \Phi(z))$$

for some smooth $\Phi : U_1 \cap U_2 \longrightarrow Hom(\mathbb{R}^k)$.

Again this is an **ismorphism** if Ψ is a diffeomorphism (exercise: show that in this case, ϕ is a diffeomorphism and Ψ has an inverse bundle map covering ϕ^{-1})

Definition 1.9. Let $f : B_1 \longrightarrow B_2$ be a smooth map and let $\pi_2 : V_2 \longrightarrow B_2$ be a smooth map. Then the **pullback bundle** $f^*\pi_2 : f^*V_2 \longrightarrow B_2$ is the bundle is defined as follows:

$$f^*V_2 \equiv \{(b,x) \in B_1 \times V_2 \,|\, f(b) = \pi_2(x)\}$$

and

$$f^*\pi_2(b,x) \equiv x.$$

(Excercise check that this is a vector bundle).

We have a natural bundle map

$$\Psi : f^*V_2 \longrightarrow V_2, \quad \Psi(b,x) = x$$

covering f.

Definition 1.10. If $B_1 \subset B_2$ is a submanifold and $\pi_2 : V_2 \longrightarrow B_2$ is a submanifold then we define the **restriction of π_2 to B_1**

$$\pi_2|_{B_1} : V_2|_{B_1} \longrightarrow B_1$$

as

$$\pi_2|_{B_1} \equiv \iota^*\pi_1, \quad V_2|_{B_1} \equiv \iota^*V_2$$

where $\iota : B_1 \hookrightarrow B_2$ is the inclusion map.

We also have other ways of producing now bundles from old ones.

Definition 1.11. Let $\pi : V \longrightarrow B, \quad \pi' : V' \longrightarrow B$ be vector bundles. We define the **direct sum**

$$\pi_1 \oplus \pi_2 : V_1 \oplus V_2 \longrightarrow B$$

to be the bundle whose fiber at $b \in B$ is the direct sum of the fibers of π_1 and π_2 at b.

More precisely: We suppose that our vector bundle π has transition data $\Phi_{ij} : U_i \cap U_j \longrightarrow GL(\mathbb{R}^k)$ coming from an open cover $(U_i)_{i\in S}$ and similarly π' has transition data $\Phi'_{ij} : U'_i \cap U'_j \longrightarrow GL(\mathbb{R}^k)$ coming from an open cover $(U'_i)_{i\in S'}$. Since the bases of these these vector bundles are the same, we can replace our open covers with refinements so that $S = S'$ and $U'_i = U_i$ for all $i \in S = S'$. (For instance we can consider the refined open cover $(U_i \cap U'_j)_{i\in S, j\in S'}$ with transition data $\Phi_{(i_1,j_1),(i_2,j_2)} \equiv \Phi_{i_1 i_2}|_{U_{i_1}\cap U'_{j_1}\cap U_{i_2}\cap U'_{j_2}}$ for all $(i_1,j_2),(i_2,j_2) \in S \times S'$ which defines π and we can do the same for π')

Then the transition data for the direct sum is just

$$\Phi_{ij} \oplus \Phi'_{ij} : U_i \cap U_j \longrightarrow GL(\mathbb{R}^k) \oplus GL(\mathbb{R}^{k'}) \subset GL(\mathbb{R}^{k+k'}).$$

Definition 1.12. We can define the **tensor product** $\pi \otimes \pi' : V \otimes V' \longrightarrow B$ of these vector bundles in a similar way by using the transition data:

$$\Phi_{ij} \otimes \Phi'_{ij} : U_i \cap U_j \longrightarrow GL(\mathbb{R}^k \otimes \mathbb{R}^{k'} = \mathbb{R}^{k_1 k_2})$$

where $\Phi_{ij} \times \Phi'_{ij}(x_1 \otimes x_2) = \Phi_{ij}(x_1) \otimes \Phi'_{ij}(x_2)$.

The **Dual** $\pi^* : V^* \longrightarrow B$ has transition data $\Phi^*_{ij} : U_i \cap U_j \longrightarrow GL((\mathbb{R}^k)^*)$.

Similarly $Hom(V_1, V_2)$ can be defined with transition data:

$$\Phi^{Hom}_{ij} : U_i \cap U \longrightarrow GL(Hom(\mathbb{R}^k, \mathbb{R}^{k'}), \quad \Phi^{Hom}_{ij}(x).(\phi) = \Phi'_{ij}(x) \circ \phi \circ \Phi_{ij}(x).$$

Or as $\pi^* \otimes \pi' : V^* \otimes V' \longrightarrow B$.

Exercise: Define the wedge product $\wedge^k V$ in a similar way.

Definition 1.13. Let $\pi : V \longrightarrow B$ be a vector bundle of dimension k and $V' \subset V$ a vector subbundle of dimension k'. The **quotient bundle** $\pi_{V/V'} : V/V' \longrightarrow B$ is defined as follows:

Let $\pi' \equiv \pi|_V$. We wish to construct these so that each fiber over $b \in B$ is the quotient vector space $\pi^{-1}(b)/(\pi')^{-1}(b)$. First of all we define this as a set and then we specify the trivializations. We define $V/V' \equiv V/\sim$ where $x \sim x'$ if and only if x and x' are in the same fiber $\pi^{-1}(b)$ of π and $[x] = [x'] \in \pi^{-1}(b)/(\pi')^{-1}(b)$. We define $\pi_{V/V'}$ to be the map sending $[x]$ to $\pi(x)$.

We will now construct the trivializations of $\pi_{V/V'}$. Choose a fine enough open cover $(U_i)_{i\in S}$ with trivializations $\tau_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^k$ so that there is a fixed subspace $H_i \subset \mathbb{R}^k$ of dimension $k-k'$ so that $\pi_{U_i}(\tau_i((\pi')^{-1}(x))) \subset \mathbb{R}^k$ is a subspace of $\mathbb{R}^k$ transverse to H_i where $\pi_{U_i} : U_i \times \mathbb{R}^k \twoheadrightarrow U_i$ is the natural projection.

For each $i \in S$ choose an isomorphism $\iota_i : \mathbb{R}^k/H_i \cong \mathbb{R}^{k'}$. Define $\Pi_i : \mathbb{R}^k \longrightarrow \mathbb{R}^{k'}$ be the composition

$$\mathbb{R}^k \twoheadrightarrow R^k/H_i \xrightarrow{\iota_i} \mathbb{R}^{k'}.$$

Now we define

$$\overline{\tau}_i : \pi^{-1}_{V/V'}(U_i) \longrightarrow U_i \times \mathbb{R}^{k'}, \quad \overline{\tau}_i([x]) \equiv \Pi_i(\tau_i(x)).$$

Exercise: show these maps are well defined and satisfy (1) and (2) from Definition 1.1.

Definition 1.14. Let $B \subset B'$ is a submanifold. The **normal bundle** of B inside B' is the vector bundle $(TB'|_B)/TB$.

Example 1.15. real projective space: Let $S^n \equiv \{x \in \mathbb{R}^{n+1} \ : \ |x| = 1 \ \}$ be the unit sphere. We define

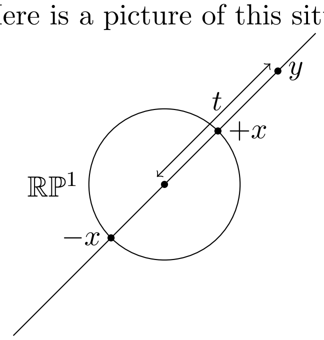
$$\mathbb{RP}^n \equiv S^n/\sim, \quad x \sim x' \text{ iff } x = \pm x'.$$

We will write elements of $\mathbb{RP}^n$ as $\{\pm x\}$ where $x \in S^n$.

Define

$$V \equiv \{(\pm x, y) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \ : \ y = tx \text{ for some } t \in \mathbb{R}.\}.$$

Here is a picture of this situation in the case $n = 1$:



We have a line bundle called $\mathcal{O}_{\mathbb{P}\mathbb{P}^n}(-1)$ defined as:

$$\pi : B \longrightarrow \mathbb{RP}^n, \quad \pi(\pm x, y) = \pm x.$$

This has trivializations defined as follows: We define $S \equiv \{0, \cdots, n\}$. We define

$$U_i \subset \mathbb{RP}^n, \quad U_i \equiv \{\pm(x_0, \cdots, x_n) \in \mathbb{RP}^n \ : \ x_i \neq 0\}.$$

We have an associated trivialization

$$\tau_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}, \quad \tau_i(\pm x, (y_0, \cdots, y_n)) \equiv (\pm x, y_i)$$

where $\mathrm{sgn}(x_i) \equiv x_i/|x_i|$.

Exercise: Check that this is a well defined map and a bijection.

We have that

$$\tau_j \circ \tau_i^{-1}(\pm(x_0, \cdots, x_n), y_i) = \frac{x_j}{x_i} y_i.$$

Hence τ_i satisfies (1) and (2) from Definition 1.1 where $\Phi_{ij}(\pm(x_0, \cdots, x_n)) = \frac{x_j}{x_i}$.

We also have other line bundles $\mathcal{O}_{\mathbb{RP}^n}(n) \equiv \mathcal{O}_{\mathbb{RP}^n}(-1)^{\otimes n}$ if $n > 0$ and $\mathcal{O}_{\mathbb{RP}^n}(0) \equiv \mathbb{RP}^n \times \mathbb{R}$ and $\mathcal{O}_{\mathbb{RP}^n}(-n) \equiv (\mathcal{O}(-1)^*)^{\otimes n}$.

Definition 1.16. A vector bundle $\pi : V \longrightarrow B$ is **trivial** if it is isomorphic to $B \times \mathbb{R}^k$. In other words, there is a bundle isomorphism $\Psi : V \longrightarrow B \times \mathbb{R}^k$. Such a bundle isomorphism is called a **global trivialization**.

Lemma 1.17. Suppose that $\pi : V \longrightarrow B$ is a trivial bundle. Then for any smooth map $f : B' \longrightarrow B$, we have that $f^*\pi : f^*V \longrightarrow B'$ is also trivial.

Proof. First of all we have a trivialization $\tau : V \longrightarrow B \times \mathbb{R}^k$. Recall that

$$f^*V \equiv \{(b', x) \in B' \times V \ : \ f(b') = \pi(x) \ \}.$$

Hence we have a natural bundle homomorphism

$$\Psi : f^*V \longrightarrow V, \quad \Psi(b', x) \equiv x.$$

Let $\pi_{\mathbb{R}} : B \times \mathbb{R}^k$ be the natural projection map. Define

$$\tau' : f^*V \longrightarrow B' \times \mathbb{R}^k, \quad \tau'(b', x) \equiv (b', \pi_{\mathbb{R}}(\tau(\Psi(b', x)))).$$

Exercise: show that τ' is a trivialization of $f^*\pi$. $\square$

We have the following immediate corollary (due to the fact that the restriction map is pullback by the inclusion map)

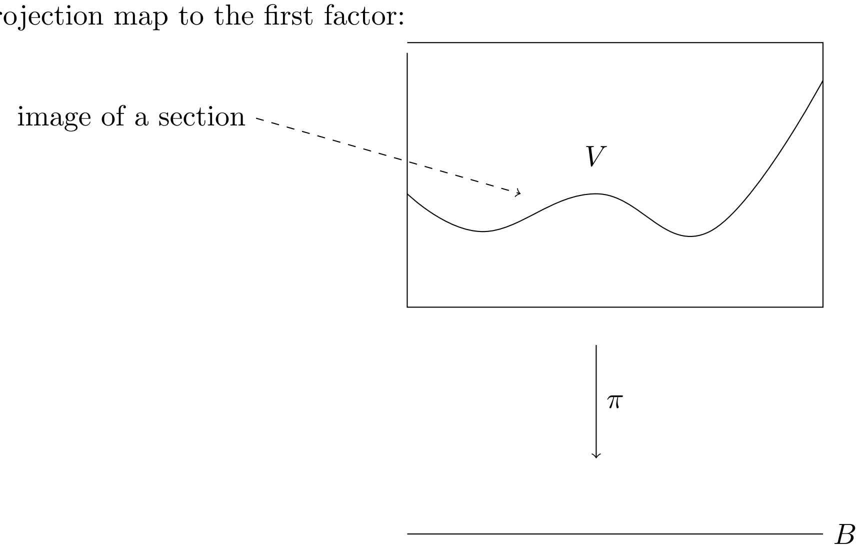
Corollary 1.18. Suppose that $\pi : V \longrightarrow B$ is a trivial bundle and $B' \subset B$ is a submanifold. Then $\pi|_{B'} : V|_{B'} \longrightarrow B'$ is a trivial bundle.

We wish to construct some non-trivial bundles. Before we do this we need another definition:

Definition 1.19. Let $\pi : V \longrightarrow B$ be a vector bundle. A **section** or **cross-section** is a smooth map $s : B \longrightarrow V$ satisfying $\pi \circ s = \mathrm{id}_B$.

The **zero section** is the section sending $b \in B$ to 0 in the vector space $\pi^{-1}(b)$(in other words, it is equal to 0 when we compose it with any trivialization τ).

Here is a picture of the image of a section in the case that $V = \mathbb{R} \times \mathbb{R}$, $B = \mathbb{R}$ and π is the projection map to the first factor:



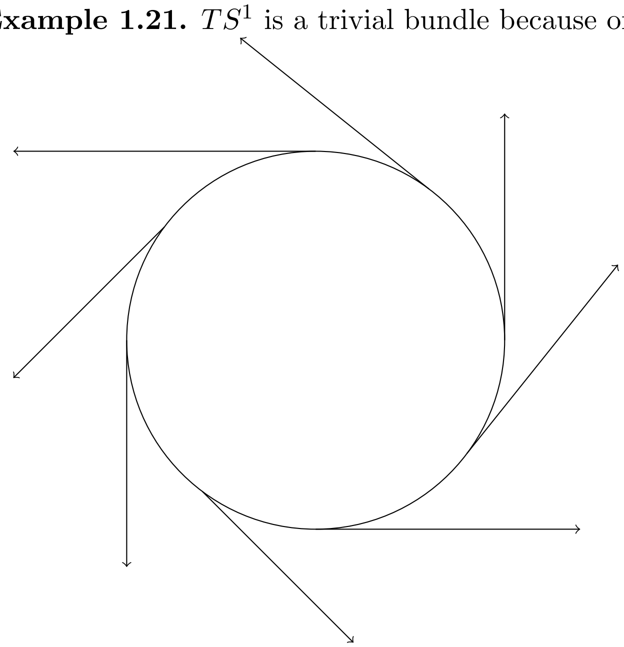
Note that a section s is uniquely determined by its image in V. This is because the image of any section is a smooth submanifold $\widetilde{B} \subset V$ so that $\pi|_{\widetilde{B}}$ is a diffeomorphism. And conversely if we have any such submanifold, then we have a section $s : B \longrightarrow V$ by defining $s(b)$ to be the unique intersection point $\pi^{-1}(b) \cap \widetilde{B}$.

Lemma 1.20. Let $\pi : V \longrightarrow B$ be a vector bundle of rank k. Then π is a trivial vector bundle if and only if k non-zero sections $s_1, \cdots, s_k$ so that $s_1(b), \cdots, s_k(b)$ form a basis of $\pi^{-1}(b)$ for all $b \in B$.

Proof. Suppose that $\tau : V \longrightarrow B \times \mathbb{R}^k$ is a trivialization. Fix a basis $e_1, \cdots, e_k$ for $\mathbb{R}^k$. Then our sections are $s_j(b) \equiv \tau^{-1}(b, e_j)$ for each $j \in \{1, \cdots, k\}$. This have the properties we want.

Conversely, suppose that we have sections $s_1, \cdots, s_k$ so that $s_1(b), \cdots, s_k(b)$ form a basis of $\pi^{-1}(b)$ for all $b \in B$. Then we define our trivialization τ as follows. For each $x \in \pi^{-1}(b)$ there is a unique $(\alpha_1(x), \cdots, \alpha_k(x)) \in \mathbb{R}^k$ so that $x = \sum_{j=1}^k \alpha_j(x) s_j(\pi(x))$. The functions $\alpha_1(x), \cdots, \alpha_k(x)$ smoothly vary as x smoothly varies due to the fact that the sections are smooth. We define $\tau(x) \equiv (\pi(x), (\alpha_1(x), \cdots, \alpha_k(x))$. This is a trivialization of π. □

Example 1.21. TS^1 is a trivial bundle because of the following picture:



A manifold is called **parallelizable** if its tangent bundle is trivial.

One can also show that the three sphere is parallelizable. Here $S^3 \subset \mathbb{R}^4$ is the unit sphere and so $TS^3 \subset T\mathbb{R}^4 \cong \mathbb{R}^4 \times \mathbb{R}^4$. The three sections forming a basis for each fiber are: $s_i(x) = (x, \overline{s}_i(x))$ where

$$\overline{s}_1(x) = (-x_2, x_1, -x_4, x_3),$$

$$\overline{s}_2(x) = (-x_3, x_4, x_1, -x_2),$$

$$\overline{s}_3(x) = (-x_4, -x_3, x_2, x_1).$$

These formulas come from the quatermionic multiplication on $\mathbb{R}^4$ [Steenrod 1951, section 8.5].

Lemma 1.22. The bundle $\mathcal{O}_{\mathbb{RP}^n}(-1)$ from Example 1.15 is not trivial.

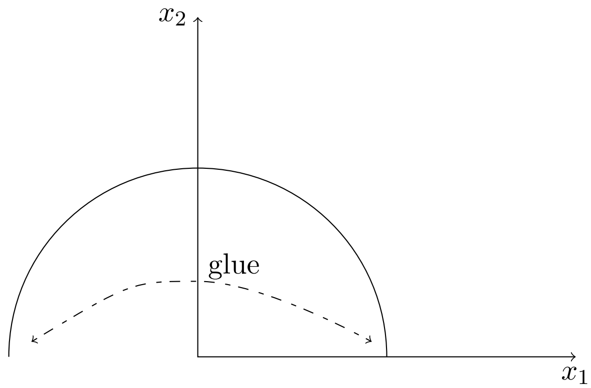
Proof. Let $\pi : V \longrightarrow \mathbb{RP}^n$ be the bundle $\mathcal{O}(-1)$ as constructed in Example 1.15.

Let

$$\iota_k : \mathbb{RP}^k \hookrightarrow \mathbb{RP}^n, \quad \iota_k((\pm(x_0, \cdots, x_k)) = (x_0, \cdots, x_k, 0, \cdots, 0)$$

be the natural embedding. In particular $\mathbb{RP}^1 \subset \mathbb{RP}^n$ is a submanifold. Therefore it is sufficient to show that $\pi|_{\mathbb{RP}^1}$ is not trivial by Corollary 1.18. By construction, $\pi|_{\mathbb{RP}^1}$ is isomorphic to $\mathcal{O}_{\mathbb{RP}^1}(-1)$. Therefore we only need to prove this when $n = 1$.

In this case $\mathbb{RP}^1$ is a semicircle with opposite ends identified as in the picture below:



This semi-circle is parameterized by the coordinate x_1. So from now on we will refer to points on this semi-circle with the coordinate x_1. The coordinate x_2 is equal to $\sqrt{1-x_1^2}$. The region U_1 is the subset of this semi-circle where $x_1 \neq 0$. This region is homeomorphic to:

$$[-1, 0) \cup (0, 1]/ \sim, \quad -1 \sim 1.$$

The region U_2 is the subset where $x_2 \neq 0$, which is the region $x_1 \neq \pm 1$ (i.e. the semi-circle minus the endpoints). Hence this is naturally diffeomorphic to $(-1, 1)$.

We have two trivializations $\tau_1 : \pi_1^{-1}(U_1) \longrightarrow U_1 \times \mathbb{R}$ and $\tau_2 : \pi_1^{-1}(U_1) \longrightarrow U_2 \times \mathbb{R}$. We have:

$$\tau_2 \circ \tau_1^{-1} : (U_1 \cap U_2) \times \mathbb{R} \longrightarrow (U_1 \cap U_2) \times \mathbb{R}, \quad \tau_2 \circ \tau_1^{-1}(x_1, y_1) = (x_1, \Psi_{12}(x_1).y_1)$$

where $\Psi_{12}(x_1)$ is the 1×1 matrix $\frac{\sqrt{1-x_1^2}}{x_1}$.

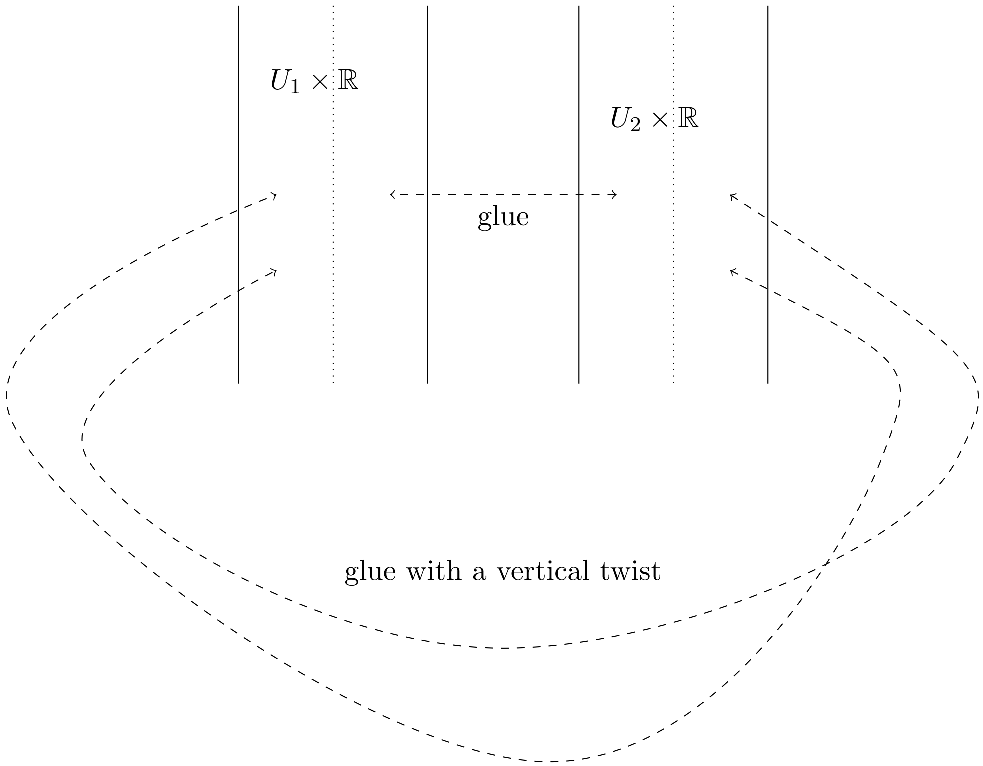
This means that V is obtained from

$$U_1 \times \mathbb{R} \cong ([-1, 0) \cup (0, 1]/ \sim) \times \mathbb{R}$$

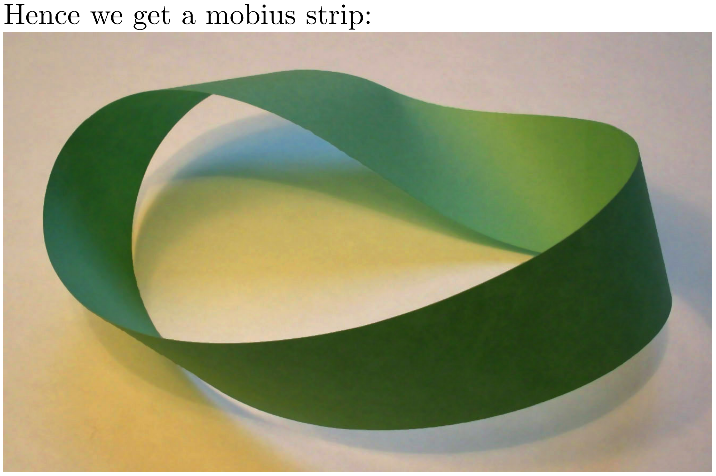
and

$$U_2 \times \mathbb{R} \cong (-1, 1) \times \mathbb{R}$$

by gluing the region $(-1, 0) \times \mathbb{R} \subset U_1 \times \mathbb{R}$ with $(-1, 0) \times \mathbb{R} \subset U_2 \times \mathbb{R}$ using a map $(x_1, y_1) \overset{(}{\longrightarrow} x_1, \Phi_{12}(x_1)y_1)$ where $\Phi_{12}(x_1) < 0$ is a negative 1×1 matrix and also gluing the region $(0, 1) \times \mathbb{R} \subset U_1 \times \mathbb{R}$ with $(0, 1) \times \mathbb{R} \subset U_2 \times \mathbb{R}$ using a map $(x_1, y_1) \overset{(}{\longrightarrow} x_1, \Phi_{12}(x_1)y_1)$ where $\Phi_{12}(x_1) > 0$ is a positive 1×1 matrix. Hence we have the following schematic picture of this gluing:

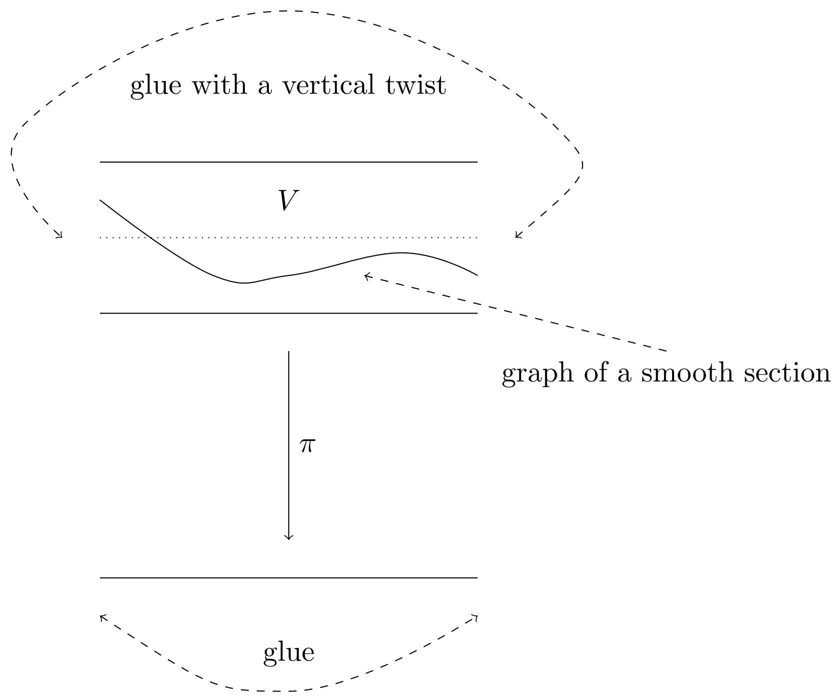


Hence we get a mobius strip:



Now if $\pi : V \longrightarrow \mathbb{RP}^1$ was a trivial bundle then it would have a nowhere zero section. But this is impossible as every section has to be zero somewhere:

Here is an illustrative diagram:



In other words, any section must cross the zero section by the intermediate value theorem. □

Euclidean Vector Bundles

Definition 1.23. Let W be a real finite dimensional vector space. Recall that a **bilinear form** is a linear map $B : W \otimes W \longrightarrow \mathbb{R}$. A **quadratic form** is a map $Q : W \longrightarrow \mathbb{R}$ satisfying $Q(v) = B(v, v)$ for some bilinear form B.

Note that we can recover the bilinear form B from Q using the formula:

$$B(v, w) = \frac{1}{2}(Q(v + w) - Q(v) - Q(w)) \tag{1}$$

Definition 1.24. A quadratic form Q is **positive definite** if $Q(v) > 0$ for all $v > 0$. Similarly a bilinear form B is **positive definite** if $Q(v) \equiv B(v, v) > 0$ for all $v \neq 0$.

A **Euclidean vector bundle** is a vector bundle $\pi V \longrightarrow B$ together with a smooth function $Q : V \longrightarrow \mathbb{R}$ whose restriction to each fiber is quadratic and positive definite. The function Q is called a **Euclidean norm**.

Equivalently by using the equation (1), a **Euclidean vector bundle** is a vector bundle $\pi : V \longrightarrow B$ together with a smooth function $\mu : V \otimes V \longrightarrow \mathbb{R}$ whose restriction to each fiber is a positive definite bilinear form. The function μ is called a **Euclidean metric**.

Exercise: show that both definitions of a Euclidean vector bundle are equivalent.

Example 1.25. V is the trivial vector bundle $B \times \mathbb{R}^k$ with Euclidean norm $(b, (x_1, \cdots, x_k)) \longrightarrow \sum_{j=1}^k x_j^2$ (or equivalently with the standard Euclidean metric given by the dot product $x_1 \otimes x_2 \longrightarrow x_1.x_2$).

Lemma 1.26. Let $\pi : V \longrightarrow B$ be a trivial vector bundle of rank k and let μ be any Euclidean metric. Then there are sections $s_1, \cdots, s_k$ which are normal and orthogonal in the sense that:

$$\mu(s_i(b) \otimes s_j(b)) = \delta_{ij}$$

for all $i, j \in \{1, \cdots, k\}$ and all $b \in B$.

Proof. By Lemma 1.20 we have k sections $s'_1, \cdots, s'_k$ so that $s'_1(b), \cdots, s'_k(b)$ form a basis for $\pi^{-1}(b)$ for each $b \in B$. We then apply the Gram-Schmidt process to these sections which results in the sections $s_1, \cdots, s_k$ that we want.

Exercise: fill in the details. □

Exercise: Show, using the above lemma, that a Euclidean vector bundle is equivalently a vector bundle with structure group $SO(k)$ [c.f. Steenrod 1951, 12.9]. (Hint: apply the above lemma to any trivialization $\tau : U \longrightarrow U \times \mathbb{R}^k$, $U \subset B$ giving us a new trivialization by Lemma 1.20.)