## 1. Pontryagin and Chern Numbers

**Definition 1.1.** Recall that a **partition** of a non-negative integer $k$ is an unordered sequence of positive numbers $I = i_1, \cdots, i_r$ whose sum is $k$. If $J = j_1, \cdots, j_s$ a partition of $l$ then we define

$$IJ \equiv i_1, \cdots, i_r, j_1, \cdots, j_s$$

to be the corresponding partition of $k + l$.

We have a partial order $\preceq$ on partitions of $k$ where $I \preceq J$ if $I$ is a **refinement of** $J$ which means that $J = I_1 I_2 \cdots I_r$ where $I_j$ is a partition of $i_j$ for all $j \in \{1, \cdots, r\}$.

**Definition 1.2.** Let $K$ be an (almost) complex manifold of complex dimension $n$ and let $I = i_1, \cdots, i_r$ be a partition of $n$. Then we define the $I$**th Chern number** to be

$$c_I([K^n]) = c_{i_1} c_{i_2} \cdots c_{i_r}[K^n] \equiv c_{i_1}(TK^n) \cup c_{i_2}(TK^n) \cup \cdots \cup c_{i_r}(TK^n)([K^n]).$$

We define $c_I([K^n]) \equiv 0$ if $I$ is a partition of an integer other than $n$.

**Example 1.3.** Recall that

$$c(\mathbb{CP}^n) = (1+u)^{n+1} = \sum_{i=0}^{n+1} \binom{n+1}{i} u^i.$$

Therefore $c_{i_1} c_{i_2} \cdots c_{i_r}[\mathbb{CP}^n] = \prod_{j=1}^r \binom{n+1}{i_j}$ if $i_1, \cdots, i_r$ is a partition of $n$.

A complex 1-manifold as one Chern number, a 2 manifold has 2 Chern numbers and in general, a complex $n$-manifold has $p(n)$ Chern numbers where $p(n)$ is the number of partitions of $n$.

Recall that $H^*(Gr_n(\mathbb{C}^\infty)) = \mathbb{Z}[\sigma_1, \cdots, \sigma_n]$ where $\sigma_j = c_j(\gamma^n_\infty)$ for all $j = 1, \cdots, n$. This means that $H^{2n}(Gr_n(\mathbb{C}^\infty))$ is the free abelian group generated by products $\prod_{j=1}^r \sigma_{i_j}$ where $i_1, \cdots, i_j$ is a partition of $n$. As a result, if $f : K \longrightarrow Gr_n(\mathbb{C}^\infty)$ is the classifying map of $TK$ then the Chern numbers of $K$ are determined by the image of the fundamental class $f_*([K^n])$ of $K$ inside $H^{2n}(Gr_n(\mathbb{C}^\infty))$ via the formula:

$$c_{i_1} c_{i_2} \cdots c_{i_r}[K^n] = \sigma_{i_1} \cdots \sigma_{i_r}([f_*(K^n)])$$

which are exactly the coefficients of $f_*([K^n])$ with respect to the basis of $H^{2n}(Gr_n(\mathbb{C}^\infty))$ as above.

**Definition 1.4.** Let $M^{4n}$ be a smooth compact oriented manifold of dimension $4n$ and let $I = i_1, \cdots, i_r$ be a partition of $n$. The $I$**th Pontryagin number** of $M^{4n}$ is

$$p_I[M^{4n}] = p_{i_1} p_{i_2} \cdots p_{i_r}[M^{4n}] \equiv p_{i_1}(TM) \cup \cdots p_{i_r}(TM)([M^{4n}]).$$

**Example 1.5.** For any partition $i_1, \cdots, i_r$ of $n$,

$$p_{i_1} p_{i_2} \cdots p_{i_r}[\mathbb{CP}^{2n}] = \binom{2n+1}{i_1}\binom{2n+1}{i_2} \cdots \binom{2n+1}{i_r}.$$

If we reverse the orientation of $M^{4n}$ then its Pontryagin classes do not change as the definition does not involve the orientation in any way, but the fundamental class changes sign. This means that if we change the orientation of $M^{4n}$ then the Pontryagin numbers $p_I[M^{4n}]$ change sign. As a result we have the following Lemma:

**Lemma 1.6.** If $M^{4n}$ has a non-zero Pontryagin number then $M^{4n}$ cannot have an orientation reversing diffeomorphism.

*Proof.* Suppose that $M^{4n}$ has an orientation reversing diffeomorphism $\tau : M \longrightarrow M$. Then if $I \equiv i_1, \cdots, i_r$ is a partition of $n$ then

$$p_I[M] = p_{i_1}(TM) \cup \cdots \cup p_{i_r}(TM)([M]) = (\tau^{-1})^* p_{i_1}(TM) \cup \cdots \cup p_{i_r}(TM)(\tau_*([M])) =$$

$$-(\tau^{-1})^* p_{i_1}(TM) \cup p_{i_r}(TM)([M]).$$

NOw since $\tau^{-1}$ is a diffeomorphism, we have that $(\tau^{-1})^*TM$ is isomorphic to $TM$ which implies that

$$(\tau^{-1})^* p_{i_1}(TM) \cup \cdots \cup (\tau^{-1})^* p_{i_1}(TM)([M]) = p_{i_1}(TM) \cup \cdots \cup p_{i_r}(TM)([M]).$$

Hence $p_I[M] = -p_I[M]$ which implies that $p_I[M] = 0$. □

**Corollary 1.7.** $\mathbb{CP}^{2n}$ admits no orientation reversing diffeomorphism.

*Proof.* We have that $p_n[\mathbb{CP}^{2n}] = \binom{2n+1}{n} \neq 0$. □

Note that $\mathbb{CP}^{2n+1}$ does have an orientation reversing diffeomorphism given by sending $[z_0, \cdots, z_{2n+1}]$ to $[\overline{z}_0, \cdots, \overline{z}_{2n+1}]$.

This is very different from the Euler class $e(M)$ since $e(S^n) \neq 0$ where $S^n$ is the $n$-sphere, yet $S^n$ admits an orientation reversing diffeomorphism.

We also have the following Lemma:

**Lemma 1.8.** If the Pontryagin number of $M^{4n}$ is non-zero then $M$ cannot be the boundary of an oriented compact $4n+1$ manifold.

*Proof.* Suppose that $M = \partial W$ is the boundary of an oriented manifold $W$ and let $\iota : M \longrightarrow W$ be the inclusion map. Let $\mu_W \in H_{4n+1}(W, M; \mathbb{Z})$ be the fundamental class. Then $\partial : H_{4n+1}(W, M; \mathbb{Z}) \longrightarrow H_{4n}(M; \mathbb{Z})$ sends $\mu_W$ to a fundamental class $\mu_M$ of $M$. Also $TW|_M = TM \oplus \mathbb{R}$ and hence $p_i(TW)|_M = p_i(TM)$. Let $\delta : H^{4n}(M; \mathbb{Z}) \longrightarrow H^{4n+1}(W, M; \mathbb{Z})$ be the natural connecting map. If $i_1, \cdots, i_r$ is a partition of $n$, then

$$p_{i_1}(TM) \cup \cdots \cup p_{i_r}(TM)(\mu_M) = p_{i_1}(TW|_M) \cup \cdots \cup p_{i_r}(TW|_M)(\partial \mu_W)$$

$$\delta(p_{i_1}(TW|_M) \cup \cdots \cup p_{i_r}(TW|_M))(\mu_W) = \delta(\iota^* p_{i_1}(TW) \cup \cdots \cup \iota^* p_{i_r}(TW))(\mu_W) = 0$$

since

$$H^{4n}(W; \mathbb{Z}) \xrightarrow{\iota^*} H^{4n}(M; \mathbb{Z}) \xrightarrow{\delta} H^{4n+1}(W, M; \mathbb{Z})$$

is a long exact sequence.

□

**Corollary 1.9.** $\mathbb{CP}^{2n}$ is not the boundary of any oriented manifold.

It turns out that $\mathbb{CP}^{2n+1}$ is the boundary of a $4n+1$ manifold as follows (sketch): We can define quaternionic projective space $\mathbb{HP}^n$ in the usual way. Identify $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}$ in the natural way. Then we have a natural quotient map $(\mathbb{C} \oplus \mathbb{C})^{n+1} \longrightarrow \mathbb{HP}^n$. This factors through the quotient map $(\mathbb{C} \oplus \mathbb{C})^{n+1} = \mathbb{C}^{2n+2} \longrightarrow \mathbb{CP}^{2n+1}$. Hence there is a natural fibration

$$\mathbb{CP}^{2n+1} \longrightarrow \mathbb{HP}^n$$

with fiber equal to $\mathbb{CP}^1 = S^2$. In particular the $S^2$ bundle can be extended to a $D^3 = \{x \in \mathbb{R}^3 \ : \ |x| \leq 1\}$ bundle and hence $\mathbb{CP}^{2n+1}$ is the boundary of a $D^3$ bundle over $\mathbb{HP}^n$.

**Definition 1.10.** Recall that the set of symmetric polynomials in $\mathbb{Z}[u_1, \cdots, u_n]$ is equal to the subalgebra $\mathbb{Z}[\sigma_1, \cdots, \sigma_n]$ freely generated by the elementary symmetric polynomials. Let $I = i_1, \cdots, i_r$ be a partition of $k$. Then by the above fact, there is a unique polynomial $s_I(\sigma_1, \cdots, \sigma_n)$ equal to $\sum_\sigma u_{\sigma(1)}^{i_1} \cdots u_{\sigma r}^{i_r}$ where we sum over all permutations $\sigma$ of $\{1, \cdots, r\}$.

If $\pi : E \longrightarrow B$ is a complex vector bundle, then we define

$$s_I(c(E)) \equiv s_I(c_1(E), \cdots, c_n(E)).$$

**Lemma 1.11.** (Thom) Let $\pi : E \longrightarrow B$, $\pi' : E' \longrightarrow B$ be complex vector bundles. Then

$$s_I(c(E \oplus E')) = \sum_{JK=I} s_J(c(E)) \cup s_K(c(E'))$$

where we sum over all partitions $J, K$ satisfying $JK = I$.

**Corollary 1.12.** If $I = k$ then

$$s_k(c(E \oplus E')) = s_k(c(E)) + s_k(c(E')).$$

*Proof.* of Lemma 1.11. Define $G \equiv Gr_n(\mathbb{C}^\infty)$, $\gamma \equiv \gamma_\infty^n$, $G' \equiv Gr_{n'}(\mathbb{C}^\infty)$ and $\gamma' \equiv \gamma_\infty^{n'}$ where $n$ is the rank of $E$ and $n'$ is the rank of $E'$. Let $f : B \longrightarrow G$ and $f' : B \longrightarrow G'$ be the classifying maps for $E$ and $E'$ respectively. Let $f \times f' : B \longrightarrow G \times G'$ be the corresponding product map.

Since $(f \times f')^* s_I(c(\gamma \times \gamma')) = s_I(c(E \oplus E'))$,

$$f^*(s_J(\gamma)) = s_J(c(E)), \quad f^*(s_K(\gamma')) = s_K(c(E'))$$

it is sufficient for us to prove that

$$s_I(\gamma \times \gamma') = \sum_{JK=I} s_J(\gamma) \otimes s_K(\gamma') \in H^*(G \times G'; \mathbb{Z}) = H^*(G; \mathbb{Z}) \otimes H^*(G'; \mathbb{Z}) = \mathbb{Z}[\sigma_1, \cdots, \sigma_n, \sigma'_1, \cdots, \sigma'_{n'}]$$

$$\subset \mathbb{Z}[u_1, \cdots, u_n, u'_1, \cdots, u'_n].$$

Here $\sigma_i$ is the $i$th symmetric polynomial in $u_1, \cdots, u_n$ and $\sigma'_i$ is the $i$th symmetric polynomial in $u'_1, \cdots, u'_{n'}$. Let $I = i_1, \cdots, i_r$. Now $\gamma \times \gamma' = p^*\gamma \oplus (p')^*\gamma'$ where $p : G \times G' \longrightarrow G$ and $p' : G \times G' \longrightarrow G'$ are the natural projection maps. Hence by the Whitney product theorem:

$$c_k(\gamma \times \gamma') = \sum_{i=0}^{k} \sigma_i \sigma'_{k-i}.$$

Therefore

$$s_I(c(\gamma \times \gamma')) = s_I\left(\sum_{i=1}^{1} \sigma_i \sigma'_{1-i}, \cdots, \sum_{i=1}^{n+n'} \sigma_i \sigma'_{n+n'-i}\right)$$

$$= \sum_{JK=I} \sum_{\sigma} \sum_{\sigma'} u_{\sigma(1)}^{j_1} \cdots u_{\sigma(s)}^{j_s} (u'_{\sigma'(1)})^{k_1} \cdots (u'_{\sigma'(t)})^{k_t}$$

where $J = j_1, \cdots, j_s$ and $K = k_1, \cdots, k_t$ satisfies $JK = I$ and we are summing over all such $JK$ and all permutations $\sigma$ of $\{1, \cdots, s\}$ and permutations $\sigma'$ of $\{frm[o]--, \cdots, t\}$.

Also

$$\sum_{JK=I} s_J(\sigma_1, \cdots, \sigma_s) s_K(\sigma'_1, \cdots, \sigma'_t)$$

is equal to the above sum. $\square$

**Definition 1.13.** If $K^n$ is a complex manifold then we define

$$s_I[K^n] \equiv s_I(c(TK))[K^n].$$

We have the following immediate corollary of Thom's lemma above.

**Corollary 1.14.**

$$s_I[K^m \times L^n] = \sum_{J,K} s_J[K^m] s_K[L^n]$$

where we now sum over all partitions $J$ of $m$ and $J$ of $n$ respectively satisfying $JK = I$.

**Corollary 1.15.**

$$s_{m+n}[K^m \times L^m] = 0.$$

**Example 1.16.** Since $c(\mathbb{CP}^n) = (1+a)^{n+1}$ where $a$ is Poincaré dual to $\mathbb{CP}^{n-1}$, we have that $c_k(\mathbb{CP}^n)$ is the $k$-th symmetric polynomial, all of whose $n+1$ entries are equal to $a$. Therefore $s_k(c(\mathbb{CP}^n)) = (n+1)a^k$. Hence $s_n[\mathbb{CP}^n] = (n+1) \neq 0$. Hence $\mathbb{CP}^n$ cannot be expressed as a product of almost complex manifolds.

We have similar formulas for Pontryagin numbers.

**Definition 1.17.** For $I = i_1, \cdots, i_r$ a partition of $k$ and $V$ real vector bundle, define

$$s_I(p(E)) \equiv s_I(p_{i_1}(V), \cdots, p_{i_r}(V)) \in H^{4k}(B; \mathbb{Z}).$$

If $M^{4n}$ is an oriented $4n$-manifold and $I = i_1, \cdots, i_r$ a partition of $n$, define

$$s_I[M^{4n}] \equiv s_I(p(TM))[M].$$

We have the following lemma which is analogous to Thom's lemma above:

**Lemma 1.18.**

$$s_I(p(E \oplus E')) = \sum_{JK=I} s_J(E) \cup s_K(E')$$

**Corollary 1.19.**

$$s_I[M^{4m} \times N^{4n}] = \sum_{J,K} s_J[M^{4m}] s_J[N^{4n}]$$

where $J$ is a partition of $m$ and $K$ is a partition of $n$ satisfying $JK = I$.

**Theorem 1.20.** (Thom) Let $K^1, \cdots, K^n$ be complex manifolds of dimension $1, \cdots, n$ respectively satisfying $s_k(c(K^k)) \neq 0$. Then the $p(n) \times p(n)$ matrix

$$c_{i_1} \cdots c_{i_r}[K^{j_1} \times \cdots \times K^{j_s}]$$

is non-degenerate where $p(n)$ is the number of partitions of $n$, $I = i_1, \cdots, i_r$ is a partition of $n$ and $J = j_1, \cdots, j_s$ is a partition of $n$.

**Theorem 1.21.** (Thom) Let $M^4, \cdots, M^{4n}$ be oriented manifolds whose satisfying $p_k(c(M^{4k})) \neq 0$. Then the $p(n) \times p(n)$ matrix

$$p_{i_1} \cdots p_{i_r}[M^{4j_1} \times \cdots \times M^{4j_s}]$$

is non-degenerate where $p(n)$ is the number of partitions of $n$, $I = i_1, \cdots, i_r$ is a partition of $n$ and $J = j_1, \cdots, j_s$ is a partition of $n$.

For example we can take $M^{4i} = \mathbb{CP}^{2i}$.

*Proof of Theorem 1.20.* Note:

$$s_I[K^{j_1} \times \cdots K^{j_s}] = \sum_{I_1 I_2 \cdots I_s = I} s_{I_1}[K^{j_1}] s_{I_2}[K^{k_2}] \cdots s_{I_s}[K^{j_s}]$$

by a generalization of Thom's lemma above. The term:

$$s_{I_1}[K^{j_1}] s_{I_2}[K^{k_2}] \cdots s_{I_s}[K^{j_s}]$$

is non-zero only when $I_q$ is a partition of $j_q$ for all $q = 1, \cdots, s$. Hence"

$$s_I[K^{j_1} \times \cdots K^{j_s}] = \sum_{I_1, \cdots, I_s} s_{I_1}[K^{j_1}] s_{I_2}[K^{k_2}] \cdots s_{I_s}[K^{j_s}]$$

where we sum over $I_1, \cdots, I_s$ where $I_q$ is a partition of $j_q$ for all $q = 1, \cdots, s$ and $I_1 I_2 \cdots I_s = I$.

This implies that if we arrange the partitions $I$ of $n$ so that they respect the ordering $\preceq$ above then

$$c_{i_1} \cdots c_{i_r}[K^{j_1} \times \cdots \times K^{j_s}]$$

becomes an upper triangular matrix. It also has non-zero diagonal entries due to the fact that $s_k(c(K^k)) \neq 0$ for all $k$ and hence must be non-degenerate. $\square$