1. Thom Spaces and Transversality

Definition 1.1. Let $\pi : E \longrightarrow B$ be a real k vector bundle with a Euclidean metric and let $E^{\geq 1}$ be the set of elements of norm ≥ 1. The **Thom space** $T(E)$ of E is the quotient $E/E^{\geq 1}$. This has a preferred basepoint $\star_E = E^{\geq 1} \in E/E^{\geq 1}$. The complement $T(E) - \star_E$ is equal to $E^{<1}$ the subset of E consisting of vectors of norm less than 1.

Note that up to homeomorphism this space does not depend on the choice of Euclidean metric on E.

Lemma 1.2. If the base B is a CW complex then $T(E)$ is a $k-1$ connected CW complex.

Proof. Let $f : D^d \longrightarrow B$ be the characteristic map for a d-cell of B. Since f^*E is trivial, we get that $f^*E^{\leq 1}$ is a product $D^d \times D^k$ which is a $d+k$ cell. Then $T(E)$ has a zero cell corresponding to $\star_E$ and then all the other cells have characteristic maps corresponding to the natural map $f^*(E^{\leq 1}) \longrightarrow T(E)$ giving us a $d+k$-cell of $T(E)$. (Exercise: fill in the details). □

Lemma 1.3. We have that $H_i(B)$ is canonically isomorphic to $H_{i+k}(T(E))$.

Proof. Since $E^{\geq 1}$ is homotopic to $E-B$, we have that this follows from the Thom isomorphism theorem. □

Definition 1.4. We let $\mathcal{C}$ be the class of all finite abelian groups. A homomorphism $h : A \longrightarrow B$ between abelian groups is called a **$\mathcal{C}$-isomorphism** if $h^{-1}(0)$ and $h(A)/B$ are in $\mathcal{C}$ (I.e. they are finite dimensional).

Theorem 1.5. Let X be a finite CW complex which is $k-1$ connected. Then the Hurewicz homomorphism

$$\pi_r(X) \longrightarrow H_r(X; \mathbb{Z})$$

is a $\mathcal{C}$-isomorphism for $r < 2k-1$.

Proof. This is true of X is an n sphere when $n \geq k$ since $\pi_r(S^n)$ is finite for $r > 2n-1$ (See Spanier).

If this theorem is true for $k-1$-connected finite CW complexes X and X' then it is true for $X \times X'$ by the Künneth formula. Hence by applying the Hurewicz theorem to the pair $(X \times X', X \vee X')$ we get

$$\pi_r(X \vee X') \cong \pi_r(X \times X') \cong \pi_r(X) \oplus \pi_r(X')$$

for $r < 2k-1$. Therefore our theorem is true for any wedge sum of spheres.

Finally let X be an arbitrary $k-1$-connected finite CW complex. Since the homotopy groups of X are finitely generated (Spanier), we can choose a finite basis for the torsion free part of $\pi_r(X)$ for each $r < 2k$ and then combine these maps into a single map

$$f : S^{r_1} \vee \cdots \vee S^{r_p} \longrightarrow X.$$

Then f is a $\mathcal{C}$-isomorphism of homotopy groups. By the Serre's generalized Whitehead theorem (Spanier page 512), this implies that it is a $\mathcal{C}$-isomorphism of homology groups as well and hence the theorem follows. □

We have the following immediate corollary.

Corollary 1.6. Let $\pi : E \longrightarrow B$ be a rank k bundle where B is a finite CW complex. Then the composition

$$\pi_{n+k}(T(E)) \longrightarrow H_{n+k}(T(E);\mathbb{Z}) \longrightarrow H_n(B;\mathbb{Z})$$

is a $\mathcal{C}$-isomorphism.

Recall that we have the following theorem:

Theorem 1.7. (**Steenrod Approximation Theorem**) Let $\pi : E \longrightarrow B$ be a smooth vector bundle with a Euclidean metric. Let σ be a continuous section of this bundle. Then for every $\delta > 0$, there is a smooth section s so that $|s(b) - \sigma(b)| < \delta$ for all $b \in B$.

We get the following corollary of this theorem:

Corollary 1.8. Let $f : M \longrightarrow N$ be a continuous map between smooth manifolds. Then f is homotopic to a smooth map.

In fact we need a stronger corollary:

Corollary 1.9. Let $f : M \longrightarrow M'$ be a continuous map between smooth manifolds and let $K \subset M$ be a closed subset and $U \subset M$ an open set whose closure is compact and disjoint from K. Then there is a continuous map $f_1 : M \longrightarrow M'$ homotopic to f so that $f_1|_U$ is smooth and $f_1|_K = f|_K$.

Proof. Choose a smooth embedding $M' \subset \mathbb{R}^N$ and let $\Psi : N \longrightarrow \mathbb{R}^n$ be a tubular neighborhood. Define

$$P : \mathrm{Im}(\Psi) \longrightarrow M', \quad P(x) = \pi_{\mathcal{N}_{\mathbb{R}^N} M'} \Psi^{-1}(x)$$

where

$$\pi_{\mathcal{N}_{\mathbb{R}^N} M'} : \mathcal{N}_{\mathbb{R}^N} M' \longrightarrow M'$$

is the normal bundle of M' in $\mathbb{R}^n$.

Let $V_1 \subset M'$ be a relatively compact open set containing $\overline{U}$. Let $\delta > 0$ be small enough so that for each $x \in V_1$, the ball of radius δ in $\mathbb{R}^n$ centered at x is contained inside $\mathrm{Im}(\Psi)$.

Recall that there is a natural $1-1$ correspondence between continuous (resp. smooth) maps $M \longrightarrow \mathbb{R}^n$ and sections of the trivial bundle $M \times \mathbb{R}^n$ given by sending a map $s : M \longrightarrow \mathbb{R}^n$ to the section $\widetilde{s} : M \longrightarrow M \times \mathbb{R}^n$, $\widetilde{s}(x) = (x, s(x))$. As a result, we can apply the Steenrod Approximation to the trivial bundle $M \times \mathbb{R}^n$ so that we get a smooth map $g : M \longrightarrow \mathbb{R}^n$ so that $|g(x) - f(x)| < \delta$ for all $x \in M$.

Now choose a smooth function $\rho : M \longrightarrow [0,1]$ so that $\rho|_U = 1$ and $\rho|_{M'-V_1} = 0$. Now define

$$g_t : M \longrightarrow M', \quad g_t(x) \equiv P\left(t\rho(x)(g(x) - f(x)) + (1 - \rho(x))f(x)\right) \quad \forall\, t \in [0,1].$$

Then g_t is a homotopy from f to $f_1 \equiv g_1$ and $f_1|_K = f|_K$ and $f_1|_U = P(g(x))|_U$ which is smooth. □

Definition 1.10. Let $f : M \longrightarrow M'$ be a smooth map and $N \subset M'$ a smooth submanifold. Let $\pi_{\mathcal{N}_{M'} N} : \mathcal{N}_{M'} N \longrightarrow N$ be the normal bundle of N inside M'. Let $Q : TM'|_N \longrightarrow \mathcal{N}_{M'} N$ be the natural quotient map. We say that f is **transverse** to N if the natural map

$$\pi_{\mathcal{N}_{M'} N} \circ Q \circ Df|_{f^*(TM'|_N)} : f^*(TM'|_N) \longrightarrow \mathcal{N}_{M'} N$$

is surjective.

Equivalently f is transverse to N if the graph $\Gamma_f \subset M \times M'$ of f is transverse to the submanifold $M \times N \subset M \times M'$.

(Exercise: show these two definitions are the same).

We will also need the following lemma:

Lemma 1.11. Let $f : M \longrightarrow M'$ be a continuous map between smooth manifolds where M is compact (possibly with boundary). Let $K \subset N$ be a compact set and $U \subset N$ an open set whose closure is disjoint from K. Then there is a smooth map $F : \mathbb{R}^k \times M \longrightarrow M'$ so that $F|_{0\times M} = f$ for some large k and a dense open subset $U \subset \mathbb{R}^k$ so that $F|_{\{x\}\times M}$ is transverse to N for all $x \in U$.

Proof. Let Γ_f be the graph of F. By the Thom Transversality theorem, there is a smooth family of submanifolds

$$\Gamma_x \subset M \times M', \quad x \in \mathbb{R}^k$$

and a dense open subset $V \subset \mathbb{R}^k$ so that Γ_x is transverse to $M \times N$ if and only if $x \in V$.

Since Γ_x is a smooth family of submanifolds, there is a small ball $B(\delta) \subset \mathbb{R}^k$ centered at 0 of radius δ so that Γ_x is transverse to $\{y\} \times M'$ for all $y \in M$ and $x \in B_\delta$. Then Γ_x is the graph of a smooth map $f_x : M \longrightarrow M'$ for all $x \in B(\delta)$. Define

$$F : B(\delta) \times M \longrightarrow M', \quad F(x,y) \equiv f_x(y)$$

and $U \equiv V \cap B(\delta)$. Then F has the properties we want after identifying the ball $B(\delta)$ diffeomorphically with $\mathbb{R}^k$. □

Theorem 1.12. Let $\pi : E \longrightarrow B$ be a smooth real oriented rank k vector bundle with a choice of Euclidean metric. Every continuous map $f : S^m \longrightarrow T(E)$ is homotopic to $g : S^m \longrightarrow T(E)$ so that

$$g|_{g^{-1}(E^{<1})} : g^{-1}(E^{<1}) \longrightarrow E^{<1}$$

is smooth and transverse to the zero section B. The oriented cobordism class of the $(m-k)$-manifold $g^{-1}(B) \subset S^m$ depends only on the homotopy class of g.

The correspondence $g \longrightarrow g^{-1}(B)$ induces a natural homomorphism from $\pi_m(T, \star_E)$ to Ω_{m-k}.

Proof. Let $\rho : [0,1) \longrightarrow [0,2)$ be a smooth map so that $\rho(x) = x$ for all $x \leq \frac{1}{2}$. Define

$$\widetilde{\rho}_t : E^{<1} \longrightarrow E^{<2}, \quad \widetilde{\rho}(v) \equiv ((1-t) + t\rho(|v|)/|v|)v$$

be a smooth family of smooth maps. We have that $\widetilde{\rho}_0$ is the natural inclusion map and $\widetilde{\rho}_1$ is a diffeomorphism.

Define $V \equiv f^{-1}(E^{<1})$. Define

$$\widetilde{f}_t : V \longrightarrow E^{<2}, \quad \widetilde{f}_t \equiv \widetilde{\rho}_t \circ f, \quad \forall t \in [0,1].$$

Let $V_1 \equiv \widetilde{f}_1^{-1}(E^{<1})$ and let $K \subset V$ be a closed set whose complement is relatively compact. By Corollary 1.9 combined with Lemma 1.11 there is a homotopy $\widetilde{g}_t : V \longrightarrow E^{<2}$ so that $\widetilde{g}_1|_U$ is smooth is smooth and transverse to B and $\widetilde{g}_t|_K = \widetilde{f}_1|_K$ for all $t \in [0,1]$.

Let

$$\widetilde{h}_t \equiv \begin{cases} \widetilde{f}_{2t} & \text{if } t \in [0,1/2] \\ \widetilde{g}_{2t-1} & \text{if } t \in [1/2,1] \end{cases}$$

be the catenation of $\widetilde{f}_t$ and $\widetilde{g}_t$. Let $Q : E^{<2} \longrightarrow T(E)$ be the natural quotient map. Finally define

$$h_t : S^m \longrightarrow T(E), \quad h_t \equiv Q \circ \widetilde{h}_t.$$

Then this is a homotopy from f to a map $g \equiv h_1$ which is smooth on $g^{-1}(E^{<1})$ and transverse to B.

We now need to show that if $h_0, h_1 : S^m \longrightarrow T(E)$ are two homotopic continuous maps which are smooth on $h_1^{-1}(E^{<1})$ and $h_2^{-1}(E^{<1})$ and which are both transverse to B then $h_1^{-1}(B)$ and $h_2^{-1}(B)$ are oriented cobordant. This is done as follows: Let $h : [0,1] \times S^m \longrightarrow T(E)$ be this continuous homotopy. Using the same techniques as above one can perturb h so that $h|_{h^{-1}(E^{<1})} : h^{-1}(E^{<1}) \longrightarrow E^{<1}$ is smooth and so that $h|_{\{0\}\times S^m} = h_0$ and $h|_{\{1\}\times S^m} = h_1$. Then $h^{-1}(B)$ is an oriented cobordism between $h_1^{-1}(B)$ and $h_2^{-1}(B)$.

Hence we get a natural map:

$$T : \pi_m(T, \star_E) \longrightarrow \Omega_{m-k}.$$

We now need to show that this is a homomorphism. Let $f_1 : S^m \longrightarrow T(E)$ and $f_2 : S^m \longrightarrow T(E)$ be continuous maps sending a marked point $\star \in S^m$ to $\star_E \in T(E)$ and so that $f_i|_{f_i^{-1}(E^{<1})}$ is smooth and transverse to B. Let $q : S^m \longrightarrow S^m \vee S^m$ be the natural quotient map collapsing the equator to the marked point $\star \in S^m \vee S^m$. We also assume that $\star$ lies on the equator of S^m. Then $f \equiv q \circ (f_1 \vee f_2) : S^m \longrightarrow T(E)$ is a continuous map so that $f|_{f^{-1}(E^{<1})}$ is smooth and transverse to 0 and $f^{-1}(B) = f_1^{-1}(B) \sqcup f_2^{-1}(B)$. Hence T is a group homomorphism. □

Theorem 1.13. (Thom) For $k > n+1$, the homotopy group

$$\pi_{n+k}(T(\widetilde{\gamma}^k_\infty), \star_{\widetilde{\gamma}^k_\infty})$$

is canonically isomorphic to the oriented cobordism group Ω_n.

Similarly

$$\pi_{n+k}(T(\gamma^k_\infty), \star_{\gamma^k_\infty})$$

is canonically isomorphic to the unoriented cobordism group of n-manifolds.

We will just focus on the oriented case. We will also only prove surjectivity (as this is sufficient for our purposes).

Lemma 1.14. If $k \geq n$ and $p \geq n$, then the natural ho homomorphism

$$T : \pi_{n+k}(T(\gamma^k_p), \star_{\gamma^k_p}) \longrightarrow \Omega_n$$

from Theorem 1.12 is surjective.

Proof. Let $[M] \in \Omega_n$ where M is an oriented n-manifold. By the Whitney embedding theorem, we have a smooth embedding $M \longrightarrow \mathbb{R}^{n+k}$. Let $\mathcal{N} \equiv \mathcal{N}_{\mathbb{R}^{n+k}} M$ be the normal bundle of M in $\mathbb{R}^{n+k}$. Choose a Euclidean metric on $\mathcal{N}$ so that we have a regularization $\Psi : \mathcal{N}^{<2} \longrightarrow \mathbb{R}^{n+k}$ (this can be done so long as the metric is small enough).

Let

$$G : \mathcal{N} \xrightarrow{\alpha} \widetilde{\gamma}^k_n \xrightarrow{\iota} \widetilde{\gamma}^k_p$$

be the composition of the Gauss map α with the natural inclusion map ι. Then we get an induced map

$$\overline{G} : T(\mathcal{N}) \longrightarrow T(\widetilde{\gamma}^k_n).$$

Let S^{n+k} be the sphere viewed as the one point compactification of $\mathbb{R}^{n+k}$. Then we have a natural quotient map

$$Q : S^{n+k} \longrightarrow S^{n+k}/(S^{n+k} - \mathrm{Im}(\Psi)) = T(\mathcal{N})$$

collapsing everything outside the tubular neighborhood of M to a point. Hence we have a natural composition:

$$f \equiv \overline{G} \circ Q : S^{n+k} \longrightarrow T(\widetilde{\gamma}^k_n).$$

Since f is smooth on $f^{-1}((\widetilde{\gamma}_p^k)^{<1})$ and since f is transverse to 0 and $f^{-1}(M) = M$ we get that the image of T contains M. $\square$

Theorem 1.15. Ω_n is finite if $n \neq 0 \bmod 4$ and has rank $p(r)$ if $n = 4r$ for some r.

Proof. From the previous theorem, Ω_n is isomorphic to $\pi_{n+k}(T(\gamma_p^k), \star_{\gamma_p^k})$ for $k, p \geq n$ which in turn is $\mathcal{C}$-isomorphic to $H_n(\widetilde{Gr}_k(\mathbb{R}^{n+p})$ by Corollary 1.6. Also $H_n(\widetilde{Gr}_k(\mathbb{R}^{n+p};\mathbb{Q})$ is freely generated by the classes

$$p_{i_1}(\widetilde{\gamma}_p^k) \cup \cdots \cup p_{i_r}(\widetilde{\gamma}_p^k), \quad i_1, \cdots, i_r \text{ is a partition of } n/4$$

if n is divisible by 4 and is 0 otherwise. This proves the theorem. $\square$

Corollary 1.16. $\Omega_* \otimes_{\mathbb{Z}} \mathbb{Q}$ is freely generated as an algebra by $[\mathbb{CP}^2], [\mathbb{CP}^4], [\mathbb{CP}^6]. \cdots$.

Proof. First of all $\Omega_n \otimes_{\mathbb{Z}} \mathbb{Q}$ is non-zero if and only if $n = 4r$ by the previous theorem. In the last lecture we showed that there is a surjection $\Omega_{4r} \longrightarrow \mathbb{Z}^{p(r)}$ which sends a manifold $[M]$ to its Pontryagin numbers and hence there is a surjection

$$\alpha_r : \Omega_{4r} \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}^{p(r)}.$$

The previous theorem tells us that the rank of $\Omega_{4r} \otimes_{\mathbb{Z}} \mathbb{Q}$ is $p(r)$ which implies that α_r is an isomorphism. This implies that $\Omega_{4r} \otimes \mathbb{Q}$ is freely generated as a vector space by products

$$\mathbb{CP}^{i_1} \times \cdots \mathbb{CP}^{i_r}, \quad i_1, \cdots, i_r \text{ is a partition of } r.$$

Hence $\Omega_* \otimes_{\mathbb{Z}} \mathbb{Q}$ is freely generated as an algebra by $[\mathbb{CP}^2], [\mathbb{CP}^4], [\mathbb{CP}^6]. \cdots$. $\square$

Corollary 1.17. Let M^n be a smooth compact oriented manifold. Then $\sqcup_{j=1}^k M^n$ is the boundary of an oriented $n+1$-manifold for some $k \geq 1$ if and only if all the Pontryagin numbers of M^n vanish.

Proof. We have that $[M^n] \in \Omega_n \otimes_{\mathbb{Z}} \mathbb{Q}$ is trivial if and only if all the Pontryagin numbers of M^n vanish by the previous theorem (since the Pontryagin numbers do not depend on the choice of representative of $[M^n]$ and since every such representative is a product of even dimensional projective spaces). Also $[M^n]$ is trivial if and only if $\sqcup_{j=1}^k M^n$ is the boundary of an oriented $n+1$-manifold for some $k \geq 1$. These two statements give us our corollary. $\square$

A Theorem of Wall actually tells us that M^n is the boundary of an oriented $n+1$-manifold if and only if all the Pontryagin numbers and Stiefel-Whitney classes vanish. This implies that Ω_n only has 2-torsion.