1. Milnor's Exotic 7-sphere

The aim of this section is to prove:

Theorem 1.1. There is a manifold M^7 homeomorphic to the 7-sphere but not diffeomorphic to the 7-sphere.

Here is some motivation for the construction of such a manifold. Recall that we can define quaternionic projective space $\mathbb{HP}^n$ in the usual way. In other words $\mathbb{HP}^n = \mathbb{H}^{n+1}/\sim$ where $\sim$ identifies $(v_0, \cdots, v_n)$ with $(kv_0, \cdots, kv_n)$ for all $k \in \mathbb{H} - 0$. Hence we have homogeneous coordinates $[v_0, \cdots, v_n] \in \mathbb{HP}^n$. This is a manifold with charts given by coordinates $[v_0, \cdots, v_{i-1}, 1, v_{i+1}, \cdots, v_n]$.

This also has a canonical bundle $\mathcal{O}_{\mathbb{HP}^n}(-1) = \mathbb{H}^{n+1} - 0$ given by the projection map

$$\pi_{\mathcal{O}} : \mathbb{H}^{n+1} - 0 \longrightarrow \mathbb{HP}^n.$$

Let us look at the case $n = 1$. Here we have that $\mathbb{HP}^1 = \mathbb{H} \cup \{\infty\}$ is diffeomorphic to S^4 (Exercise - construct an explicit diffeomorphism). There are two charts $\mathbb{H}_1 = \mathbb{H} \subset \mathbb{HP}^1$ with homogeneous coordinates $[v_0, 1]$ and $\mathbb{H}_2 = (\mathbb{H} - 0) \cup \infty \subset \mathbb{HP}^1$ with coordinates $[1, v_1]$. The chart changing map sends $v_0 \in \mathbb{H}_1 - 0$ to $v_1 = 1/v_0 \in \mathbb{H}_1 - 0$. Also the bundle $\mathcal{O}_{\mathbb{HP}^1}(-1)$ has two trivializations over these charts with associated transition map

$$\Phi_{12} : \mathbb{H}_1 - 0 \longrightarrow GL(4, \mathbb{R})_+, \quad \Phi_{12}(v_0) \cdot h = hv_0^{-1}.$$

Let $U\mathbb{H} = S^3$ be the quaternions of unit norm (i.e. $U\mathbb{H} \equiv \{x + yi + zj + wk \in \mathbb{H} \ : x^2 + y^2 + z^2 + w^2 = 1\}$). The restriction of $\pi_{\mathcal{O}}$ to S^7 is an S^3-bundle over $S^4 = \mathbb{HP}^1$.

When we looked at one dimensional bundles over $\mathbb{RP}^1$ or one dimensional complex bundles over $\mathbb{CP}^1$ we saw that they were all obtained by taking powers of Φ_{12}. The quaternions though, are more complicated since they are not commutative. As a result we have many different possibilities for the map Φ_{12}.

For instance we can define an oriented $\mathbb{R}^4 = \mathbb{H}$ bundle over $\mathbb{HP}^1$ where the transition map as above is

$$\Phi_{12}^{j,k} : \mathbb{H}_1 - 0 \longrightarrow GL(4, \mathbb{R})_+, \quad \Phi_{12}(v_0) \cdot h = v_0^k h v_0^j.$$

We will write $V_{k,j}$ for this oriented $\mathbb{R}^4$ bundle over $S^4 = \mathbb{HP}^1$.

Definition 1.2. Let $\pi : V \longrightarrow B$ be a real vector bundle. The **associated sphere bundle** SV is the set of vectors of norm 1 after choosing some metric. The **associated disk bundle** DV is the set of vectors of norm ≤ 1.

This does not depend on the choice of metric. Also $\partial DV = SV$.

The associated sphere bundle of $\mathcal{O}_{\mathbb{HP}^1}(-1)$ is S^7 since we can choose the Euclidean metric to be the standard one on the fibers viewed as subspaces of $\mathbb{H}^2$. Are the sphere bundles $SV_{j,k}$ all different?

We now need to find out which bundles have unit sphere bundles that are homeomorphic to the 7-sphere.

Lemma 1.3. Let M be a smooth compact manifold admitting a Morse function with only two critical points. Then M is homeomorphic to a sphere.

Proof. It is sufficient for us to show that M is equal to $\mathbb{D}^n \sqcup \mathbb{D}^n / \sim$ where $\sim$ identifies $\partial \mathbb{D}^n$ in the first factor with $\partial \mathbb{D}^n$ in the second factor via some diffeomorphism. This is because $M - \star$ where $\star = 0 \in \mathbb{D}^n \sqcup \emptyset$ is diffeomorphic to $\mathbb{R}^n$ which implies that M is homeomorphic to the one point compactification of $\mathbb{R}^n$ which is the sphere.

Fix a metric on M. Let $f : M \longrightarrow \mathbb{R}$ be a Morse function with two critical points one maximum and one minimum. Then the maximum of f looks like $c - \sum_i x_i^2$ in some chart. Let $D_1 \equiv \{\sum_i x_i^2 \leq 1\}$.

The minimum of f looks like $c' + \sum_i y_i^2$ in some chart. Let $D_2 \equiv \{\sum_i y_i^2 \leq 1\}$.

Define $g : \partial D_2 \longrightarrow (0, \infty)$ send $x \in \partial D_2$ to the amount of time it takes to flow x to ∂D_1 along the vector field ∇f. Let ϕ_t be the flow of ∇f. Define $\Phi : \partial D_2 \times [0,1] \longrightarrow M, \quad \Phi(x,t) \equiv \phi_{g(x)t}(x)$. Then M is equal to $D_1 \sqcup D_2 \sqcup (\partial D_2 \times [0,1])$ where ∂D_2 is identified with $\partial D_2 \times \{0\}$ and $\partial D_2 \times \{1\}$ is identified with ∂D_1 using the flow of ∇_f (i.e. the diffeomorphism $x \longrightarrow \phi_{g(x)}(x)$). Hence M is a union of two balls $D_2 \cup \Phi(\partial D_2 \times [0,1])$ and D_1 glued along their boundary via a diffeomorphism. $\square$

Lemma 1.4. If $k + j = -1$ then $SV_{k,j}$ admits a Morse function with only two critical points.

Proof. We can define a metric $\|\cdot\|$ on $V_{k,j}$ as follows. In the chart $\mathbb{H}_1 \times \mathbb{H}$ we get that $\|(v_0, x)\| \equiv |x|\sqrt{1 + |v_0|^2}$ and in the chart $\mathbb{H}_1 \times \mathbb{H}$ we get that $\|(v_1, x')\| \equiv |x'|\sqrt{1 + |v_1|^2}$. Exercise: show that this is a well defined metric (use the fact that $|uvu^{-1}| = |v|$ for all $u, v \in \mathbb{H}^2$). Let $SV_{k,j}$ be the unit disk bundle with respect to this metric.

Our Morse function $f : SV_{j,k} \longrightarrow \mathbb{R}$ is $\mathrm{Re}(x)$ for $(v_0, x) \in \mathbb{H}_1 \times \mathbb{H}$ and $\mathrm{Re}(x'v_1)$ for $(v_1, x') \in \mathbb{H}_2 \times \mathbb{H}$. Exercise: show that this map is well defined if $k + j = -1$ (this boils down to the fact that $\mathrm{Re}(uvu^{-1}) = \mathrm{Re}(v)$.

Exercise: it only has two critical points and is a Morse function. One way of proving this is to first show that in the case when $k = 0, j = -1$ (which is the bundle $\mathcal{O}_{\mathbb{HP}^1}(-1)$), the above function is $\mathrm{Re}(v_1)$ for the coordinates $(v_0, v_1) \in S^7 = SV_{0,-1} \subset \mathbb{H}^2$. This has two critical points. The key point is that for any other k, j satisfying $k + j = -1$, we have exactly the same function inside each chart and hence f also only has two critical points! $\square$

Corollary 1.5. If $k + j = 1$ then $SV_{j,k}$ is homeomorphic to S^7.

We now need to show that $V_{k,j}$ is exotic for some j, k satisfying $k + j = 1$. The key idea is to show that if it was not exotic then the signature some smooth compact 8-manifold would not be an integer by the Hirzebruch signature theorem.

Before we do this we need some preliminary Lemmas.

First of all we will compute the Pontryagin and Euler class of these bundles. Before we do this, we need some preliminary lemmas.

Lemma 1.6. Let $(\Phi_{12}^{j,k})_* : \pi_3(\mathbb{H}_1 - 0) = \pi_3(S^4) = \mathbb{Z} \longrightarrow \pi_3(GL(4,\mathbb{R})_+) = \mathbb{Z} \oplus \mathbb{Z}$ be the natural map. Then the map $\alpha : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2, \quad (j,k) \longrightarrow (\Phi_{12}^{j,k})_*(1)$ is a group homomorphism.

Proof. First of all,

$$\Phi_{12}^{j,k})(v_0) = (\Phi_{12}^{1,0})(v_0)^k \cdot (\Phi_{12}^{0,1})(v_0)^j. \qquad (1)$$

Therefore since $\mathbb{H} - 0$ is homotopic to $U\mathbb{H} = S^4$ it is sufficient for us to show that if

$$\Phi : S^3 \longrightarrow \pi_3(GL(4,\mathbb{R})_+), \quad \Phi' : S^3 \longrightarrow \pi_3(GL(4,\mathbb{R})_+)$$

are two smooth maps and

$$\Phi \cdot \Phi' : S^3 \longrightarrow \pi_3(GL(4,\mathbb{R})_+), \quad \Phi \cdot \Phi'(v_0) \equiv \Phi(v_0) \cdot \Phi'(v_0)$$

is their product then

$$[\Phi \cdot \Phi'] = [\Phi] + [\Phi'] \in \pi_4(GL(4, \mathbb{R})_+).$$

To prove (1), note that S^3 is a union of two 3-balls D_+, D_- with boundary along the equator. We can homotope Φ and Φ' so that $\Phi|_{D_-} = id$ and $\Phi'|_{D_+} = id$. Then $[\Phi \cdot \Phi'] = [\Phi] + [\Phi'] \in \pi_4(GL(4; \mathbb{R})_+)$. □

Corollary 1.7. The maps

$$\mathbb{Z} \oplus \mathbb{Z} \longrightarrow H^4(S^4), \quad (j, k) \longrightarrow p_1(V_{j,k})$$

$$\mathbb{Z} \oplus \mathbb{Z} \longrightarrow H^4(S^4), \quad (j, k) \longrightarrow e(V_{j,k})$$

are group homomorphisms.

Proof. By the long exact sequence of the $GL(4; \mathbb{R})$-principal bundle we get that

$$\pi_3(GL(4, \mathbb{R})) = \pi_4(\widetilde{GL}_4(\mathbb{R}^\infty). \tag{2}$$

The classifying maps $f_{j,k} : S^4 \longrightarrow \widetilde{GL}_4(\mathbb{R}^\infty)$ represent elements $A_{j,k} \in \pi_4(\widetilde{GL}_4(\mathbb{R}^\infty))$ corresponding to the elements $\alpha(j, k)$ from Lemma 2 under the isomorphism (2). Hence we get our group homomorphism By Lemma 1.6. □

Lemma 1.8. The Pontryagin class of $V_{j,k}$ is $(k - j) \in \mathbb{Z} = H^4(S^4; \mathbb{Z})$. The Euler class of $V_{j,k}$ is $j - k \in \mathbb{Z} = H^4(S^4; \mathbb{Z})$.

Proof. By Corollary 1.7, the group homomorphism:

$$\beta : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H^4(S^4) = \mathbb{Z}, \quad (j, k) \longrightarrow p_1(V_{j,k})$$

are additive. Since reversing orientation does not change the Pontryagin class, we get that β is a linear function of $k - j$ and hence is equal to $d(k - j)$ for some $d \in \mathbb{Z}$.

To calculate d we just need to compute the Pontryagin class of $V_{0,1}$. Since this is a complex bundle the Pontryagin class is minus the second Chern class which in turn is equal to minus the Euler class. To compute the Euler class we will construct a section of $V_{0,1}$ transverse to zero then the Poincaré dual of its zero set will be the Euler class. The section is equal to 1 in the trivialization $\mathbb{H}_1 \times \mathbb{H}$ and v_1 in the trivialization $\mathbb{H}_2 \times \mathbb{H}$. This intersects the zero section positively once and hence the Euler class is 1. This implies that $d = -1$. The Euler class is $-(k - j) = j - k$. □

Recall that for any compact 8-manifold B, we have

$$\sigma(B) = \frac{1}{45}(7p_2[B] - p_1^2[B]).$$

Therefore

$$p_1^2[B] = 7p_2[B] - 45\sigma(B)$$

and hence

$$p_1^2[B] = -3\sigma(B) \mod 7. \tag{3}$$

Lemma 1.9. Then $p_1(DV_{j,k}) = k - j \in \mathbb{Z} \cong H^4(V_{j,k}; \mathbb{Z})$.

Proof. Since $DV_{j,k}$ is a disk bundle over S^4, we have that $H^4(V_{j,k}; \mathbb{Z}) \cong \mathbb{Z}$. Now $TS^4 \oplus \mathbb{R}$ is trivial which implies that $p(TS^4)p(\mathbb{R}) = p(TS^4 \oplus \mathbb{R}) = 1$ which implies that $p_1(TS^4) = 0 \in H_4(S^4; \mathbb{Z}) \cong \mathbb{Z}$. Also $p_1(V_{j,k}) = k - j$. Since $TDV_{j,k}|_{S^4} = TS^4 \oplus V_{j,k}$ and so $p_1(TDV_{j,k}) = p_1(TS^4) + p_1(V_{j,k}) = k - j \in \mathbb{Z} = H^4(V_{j,k;\mathbb{Z}})$. □

Proof of Theorem 1.1. Choose $k = 1$ and $j = -2$. Since $k + j = -1$, we have that $SV_{j,k}$ is homeomorphic to S^7 by Corollary 1.5.

Suppose (for a contradiction) that $SV_{j,k}$ is diffeomorphic to S^7. Let D^8 be the unit ball of dimension 8. We construct an 8 manifold $E \equiv DV_{j,k} \sqcup D^8 / \sim$ where $\sim$ identifies $\partial V_{j,k} = SV_{j,k}$ with $\partial D^8 = S^7$ via our assumed diffeomorphism $SV_{j,k} \leq S^7$.

Since $H^4(B;\mathbb{Z}) = H^4(DV_{j,k};\mathbb{Z}) \cong \mathbb{Z}$ we have that $p_1(TB) = p_1(TDV_{j,k}) = k - j = -3 \in \mathbb{Z} \cong H^4(B;\mathbb{Z})$ by Lemma 1.9.

Also since the normal bundle of $S^4 \subset DV_{j,k} \subset B$ is $V_{j,k}$ we get that the self intersection of S^4 is equal to the Euler number of $V_{j,k}$ which is $j - k = -3$ by lemma 1.8. Hence $p_1^2[B]$ is $k - j = 3$ times the self intersection number of S^4 which is $-(k-j)^2 = -9$.

The signature of B is -1 since the self intersection of S^4 is $j - k = -3 < 0$.

We have that B cannot be a smooth manifold since $\sigma(B) = -1 \neq -9 \mod 7$ contradicting Equation (3). Hence $V_{j,k}$ is not diffeomorphic to S^7.

□