1. A Review of Cohomology of Manifolds

Good texts include Spanier and Hatcher.

Definition 1.1. The standard n-simlex is the convex set $\Delta^n \subset \mathbb{R}^{n+1}$ given by the set of $(n+1)$-tuples $(t_0, \cdots, t_n)$, satisfying $\sum_{i=1}^{n+1} t_i = 1$.

A continuous map from Δ^n to a topological space X is called a **singular n-simplex.** The **ith face of a singular n-simplex** $\sigma : \Delta^n \leq X$ is the $n-1$-simplex

$$\sigma \circ \phi_i : \Delta^{n-1} \longrightarrow X$$

where

$$\phi_i : \Delta^{n-1} \longrightarrow x, \quad \phi_i(t_0, \cdots, t_{n-1}) \equiv \phi(t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_{n-1}).$$

For each $n \geq 0$, the **singular chain group** $C_n(X, \Lambda)$ where Λ is a commutative ring, is the free Λ-module generated by the set of singular n-simplices. In other words, it is the set of formal finite linear combinations $\sum_{i \in S} a_i[\sigma_i]$ where $|S|$ is finite and $a_i \in \Lambda$ for all $i \in S$ and $(\sigma_i)_{i \in S}$ are singular n-simplices. For $n < 0$, $C_n(X, \Lambda)$ is defined to be 0.

The **boundary homomorphism** is the map

$$\partial : C_n(X, \Lambda) \longrightarrow C_{n-1}(X, \Lambda), \quad \partial(\sum_{i \in S} a_i \sigma_i) = \sum_{i \in S} a_i \sum_{j=0}^{n} [\sigma_i \circ \phi_i].$$

Then $\partial \circ \partial = 0$. We define $Z_n(X, \Lambda) \equiv \ker(\partial)$ and $B_n(X, \Lambda) \equiv \mathrm{Im}(\partial)$ and $H_n(X, \Lambda) \equiv Z_n(X, \Lambda)/B_n(X, \Lambda)$ to be the **nth singular homology gorup of X**. Note we call this a group, but it is really a Λ-module (it is a group in the case when $\Lambda = \mathbb{Z}$ as abelian groups are $\mathbb{Z}$-modules).

The **cochain group** is defined to be the dual of the singular chain group:

$$C^n(X, \Lambda) \equiv Hom_\Lambda(C_n(X, \Lambda), \Lambda).$$

In other words, it is the set of maps:

$$s : \mathbb{C}_n(X, \Lambda) \longrightarrow \Lambda$$

satisfying

$$s(\sum_{i \in S} a_i[\sigma_i]) = \sum_{i \in S} a_i s([\sigma_i])$$

for each finite formal sum $\sum_{i \in S} a_i[\sigma_i]$ as above.

The **coboundary map** is the Λ-linear map

$$\delta : C^n(X; \Lambda) \longrightarrow C^{n+1}(X, \Lambda), \quad \delta(s)(x) \equiv (-1)^{n+1} s(\partial(x)), \quad \forall\, s \in C^n(X; \Lambda),\ x \in C_{n+1}(X; \Lambda).$$

Again $Z^n(X; \Lambda) \equiv \ker(\delta)$ and $B^n(X; \Lambda) \equiv \mathrm{Im}(\delta)$ and $H^n(X; \Lambda) \equiv Z^n(X; \Lambda)/B^n(X; \Lambda)$ is the **nth cohomology group of X**.

Note that we have not used the multiplicative structure of Λ (yet). Hence, one could generalize our coefficient system to include Λ-modules for instance. This is useful in obstruction theory for instance. We will stick to having coefficients in a commutative ring Λ.

If $A \subset X$ is a topological subspace, we have groups $C_n(X, A; \Lambda) \equiv C_n(X; \Lambda)/C_n(A; \Lambda)$ and hence we can define $H_n(X, A; \Lambda)$ in the usual way using the induced boundary homomorphism ∂ (since ∂ maps the submodule $C_n(A; \Lambda)$ to itself). We define $C^n(X, A; \Lambda) \equiv \ker(C^n(X; \Lambda) \longrightarrow C^n(A; \Lambda)$. Again δ maps the submodule $C^n(X, A; \Lambda)$ to itself and hence we can define $H^n(X, A; \Lambda)$.

Relationship between homology and cohomology.

From now on we will assume that Λ is a principal ideal domain (e.g a field or $\mathbb{Z}$). To simplyfy notation we will omit Λ, and just write $C_n(X), Z_n(X), H_n(X)$, etc instead of $C_n(X;\Lambda), Z_n(X;\Lambda)$, etc. Also we will write $H_*(X)$ for the sequence of groups $H_0(X), H_1(X), \cdots$.s

Theorem 1.2. Suppose that $H_{n-1}(X)$ is 0 or (more generally) a free Λ-module. Then $H^n(X)$ is canonically isomorphic $Hom_\Lambda(H_n(X);\Lambda)$. We have a similar statement for pairs (X, A).

E.g. the identity $H^n(X) = Hom_\Lambda(H_n(X);\Lambda)$ is always true when Λ is a field.

The proof is contained in Milnor and Stasheff (Appendix A) We will just explain what the canonical map $k : H^n(X) \longrightarrow Hom_\Lambda(H_n(X);\Lambda)$ is.

Given elements $x \in H^n(X)$ and $\xi \in H_n(X)$ define the **Kronecker index** $< x, \xi > \in \Lambda$ as follows: Choose a representative $\widetilde{x} \in C^n(X;\Lambda)$ of x and a representative $\widetilde{\xi} \in C_n(X;\Lambda)$ of ξ. Then we define $< x, \xi > \equiv \xi(x)$.

Exercise: show that this does not depend on the choice of representatives $\widetilde{x}, \widetilde{\xi}$ of x and ξ repsectively.

Hence we get a natural map:

$$k : H^n(X) \longrightarrow Hom_\Lambda(H_n(X);\Lambda), \quad k(x)(\xi) \equiv < x, \xi > .$$

Homology of a CW complex.

Recall that a CW complex is a topological space obtained by starting with 0 dimensional balls, then gluing 1-balls to the 0 balls along the boundary of these 1-balls giving us a 1-skeleton, and then gluing 2-balls to the 1-skeleton along their boundary giving us the 2-skeleton etc....

Let K be the underlying topological space of a CW complex and let $K^n \subset K$ be its n-skeleton.

Lemma 1.3. $H_i(K^n, K^{n-1}) = 0$ for all $i \neq n$ and is a free Λ-module generated by the set of n-cells (that are glued) when $i = n$

Proof. Let $S \subset K^n$ be a finite set with exactly one point in the interior of each n-cell and no other points. Since K^{n-1} is a deformation retract $K^n - S$ and hence of a neighborhood of itself inside K^n, we get that $H_i(K^n, K^{n-1}) = \widetilde{H}_i(K^n/K^{n-1})$. We have that K^n/K^{n-1} is homeomorphic to a wedge sum of spheres giving us our result. □

Corollary 1.4. $H_i K^n$ is zero for $i > n$ and free of rank the number of cells when $i = n$ and isomorphic to $H_i(K)$ for $i < n$.

Proof. This is true when $n = 0$ since K^0 is a disjoint union of points corresponding to the number of 0-cells. Now suppose our lemma is true for the $n-1$ skeleton and consider the n skeleton. Consider long exact sequence:

$$H_i(K^{n-1}) \longrightarrow H_i(K^n) \longrightarrow H_i(K^n, K^{n-1}).$$

The condition that $H_i(K^n, K^{n-1}) = 0$ for $i < n$ implies that $H_i(K^{n-1}) \longrightarrow H_i(K^n)$ is an isomorphism for all $i < n-1$ by our induction hypothesis. If K is infinite dimensional then one needs to use the fact that its homology is the direct limit of the homology of K^n as n goes to infinity. □

Definition 1.5. The free module $H_*(K^n, K^{n-1})$ is called the n**th chain group of the the CW complex** K. We will write $\mathcal{C}_n K$ for this module. Similarly $\mathcal{C}^n(K) \equiv Hom_\Lambda(\mathcal{C}_n K, \Lambda)$ is the n**th cochain group of the CW complex** K.

The **boundary** is the natural map $\partial_n : \mathcal{C}_{n+1}K \longrightarrow \mathcal{C}_n K$ coming from the long exact sequence of the triple (K^{n+1}, K^n, K^{n-1}):

$$H_{n+1}(K^{n+1}, K^n) \longrightarrow H_n(K^n, K^{n-1}) \longrightarrow H_n(K^{n+1}, K^{n-1}) \longrightarrow H_n(K^{n+1}, K^n).$$

Similarly one can define the coboundary map.

Theorem 1.6. The (co)homology of the CW chain complex of K is canonically isomorphic to its singular (co)homology.

The canonical homomorp-hism comes from the long exact sequence:

$$0 \longrightarrow H_n(K^n, K^{n-2}) \longrightarrow \mathcal{C}_n K \longrightarrow \mathcal{C}_{n-1} K$$

coming from the triple (K^n, K^{n-1}, K^{n-2}) combined with the fact that $H_n(K^n, K^{n-2}) = H_n(K^n)$ from Lemma 1.4. See Milnor's book.

cup product:

Let $\sigma : \Delta^{m+n} \longrightarrow X$ be a singular simplex. The **front** m**-face** of sigma, is the singular simplex $\sigma \circ \alpha_m$ where

$$\alpha_m : \Delta^m \longrightarrow \Delta^{m+n}, \quad (t_0, \cdots, t_m) \longrightarrow (t_0, \cdots, t_m, 0, \cdots, 0).$$

The **back** n**-face of** σ is the composition $\sigma \circ \beta_n$ where

$$\beta_n : \Delta^n \longrightarrow \Delta^{n+m}, \quad (t_0, \cdots, t_n) \longrightarrow (0, \cdots, 0, t_0, \cdots, t_n).$$

The **cup product** $c \cup c' \in C^{m+n}(X)$ of $c \in C^m(X), c' \in C^n(X)$ is defined as:

$$c \cup c'([\sigma]) \equiv (-1)^{mn} c([\sigma \circ \alpha_m]) c'([\sigma \circ \beta_n]) \in \Lambda.$$

Then

$$\delta(c \cup c') = (\delta c) \cup c' + (-1)^m c \cup (\delta c').$$

Hence the cup product descends to a product

$$\cup : H^m(X) \otimes H^n(X) \longrightarrow H^{m+n}(X).$$

This is **graded commutative** in the sense that $[c] \cup [c'] = (-1)^{mn} [c'] \cup [c]$. Hence $H^*(X)$ is a **graded commutative ring**.

If $A \subset X$ and $B \subset X$ are relatively open when considered as open subsets of $A \cup B$. Then one has a cup product map:

$$H^m(X; A) \otimes H^n(X; B) \longrightarrow H^{m+n}(X; A \cup B).$$

Künneth formula:

Definition 1.7. Let $p_1 : X \times Y \longrightarrow X$ and $p_2 : X \times Y \longrightarrow Y$ be the natural projection maps. The **cross product map** (or **external product**) is the map:

$$\times : H^m(X) \otimes H^n(Y) \longrightarrow H^{m+n}(X \times Y), \quad x \times y \equiv (p_1^* x) \cup (p_2^* y).$$

Similarly this can be defined for pairs:

$$\times : H^m(X, A) \otimes H^n(Y, B) \longrightarrow H^{m+n}(X \times Y, (A \times Y) \cup (X \times B)).$$

Theorem 1.8. Let X, Y be CW complexes such that each $H^i(X)$ is a torsion free Λ module (e.g. when Λ is a field) and Y only has finitely many cells in each dimension. Then the cross product map above is an isomorphism.

homology of manifolds.

Lemma 1.9. Let M be a smooth manifold. Then $H_n(M; M - x) = \Lambda$ and $H_i(M; M - x) = 0$ for all $i \neq n$ for all $x \in M$.

This is done by excision. I.e. $H_n(M, M-x) = H_n(\mathbb{R}^n, \mathbb{R}^n - x)$.

Definition 1.10. An **orientation** on M is a choice $\mu_x \in H_n(M, M-x) - 0$ for each $x \in M$ so that μ_x 'varies continuously' with respect to x.

I.e. For each $x \in M$, there is a relatively compact neighborhood $N \ni x$ and a class $\mu_N \in H_n(M, M-N)$ so that the image of μ_N in $H_n(M, M-y)$ is μ_y for each $y \in N$.

A manifold with orientation is called an **orientated manifold**. If a manifold admits an orientation then we call it **orientable.**

The following Lemma is **important.**

Lemma 1.11. Let M be an oriented manifold with orientation $(\mu_x)_{x\in M}$. For each compact $K \subset M$, there is a **unique** $\mu_K \in H^n(M, M-K)$ which maps to μ_y for each $y \in K$.

The uniqueness part is not too difficult. The existence part is difficult. See Milnor Appendix A Lemma A.7. The key idea is to construct μ_{K_i} for compact $(K_i)_{i\in I}$ which are contractible with non-empty interiors covering M and then 'glue' together these μ_{K_i}'s together.

Definition 1.12. If M is a compact oriented manifold with orientiation $(\mu_x)_{x\in M}$ then its **fundamental class** $[M] \in H_n(M)$ is the unique class whose restriction to $H_n(M, M-x)$ is μ_x for all $x \in M$.

cohomology with compact support.

Definition 1.13. A cochain c has **compact support** in X if there is a compact set $K \subset X$ so that $c([\sigma]) = 0$ for each $\sigma : \Delta \longrightarrow M - K \subset M$. In other words, c belongs to $C^i(X, X-K) \subset C^i(X)$. The cochains with compact support form a Λ submodule $C^i_c(X;\Lambda) \subset C^i(X;\lambda)$. Hence we have a cohomology group $C^i_c(X;\Lambda) = C^i_c(X)$.

Note if X is compact then $H^i_c(X) = H^i(X)$.

Definition 1.14. If M is an oriented n-manifold with orientation $(\mu_x)_{x\in M}$ then we have an **integration map**

$$\int : H^n_c(M) \longrightarrow \Lambda, \quad [c] \longrightarrow c(\mu_K)$$

where $K \subset M$ is a compact set containing the support of the cochain c representing a class in $H^n_c(M)$ and $\mu_K \in H_n(M, M-K)$ is the class from Lemma 1.11.

Exercise: show that this does not depend on K or the choice of representative c of the homology class $[c] \in H^n_c(M)$.

Note that in de Rham cohomology, we have $\Lambda = \mathbb{R}$ and the orientation corresponds to a volume form, and the integration map in this definition corresponds exactly to integration with respect ot the chose volume form.

Cap product operation:

Definition 1.15. We have the following map called the **cap product**:

$$\cap : C^i(X) \otimes C_k(X) \longrightarrow C_{k-i}(X), \quad a \otimes \sigma \longrightarrow a \cap \sigma \equiv (-1)^{i(k-i)}(a([\sigma \circ \beta_i]))\sigma \circ \alpha_{k-i}$$

where $\sigma \circ \beta_i$ is the back i-face of σ and $\sigma \circ \alpha_{k-i}$ is the front $k-i$ face of σ.

We have the following identities:

$$(b \cup c) \cap \xi = b \cap (c \cap \xi)$$
$$1 \cap \xi = \xi$$
$$\partial(b \cap \xi) = (\delta b) \cap \xi + (-1)^{dimb} b \cap (\partial \xi). \tag{1}$$

Definition 1.16. By Equation 1, we get that the cap product descends to a map

$$\cap : H^i(X) \otimes H_k(X) \longrightarrow H_{k-i}(X)$$

which we call the **cap product**.

Theorem 1.17. Suppose that M is oriented and compact then the map $H^i(M) \longrightarrow H_{n-i}(M)$ is an isomorphism under the map $a \longrightarrow a \cap [M]$ where $[M]$ is Defined in Definition 1.12.

Theorem 1.18. Suppose that M is oriented and not-necessarily compact then the we have an isomorphism $D : H_c^i(M) \longrightarrow H_{n-i}(M)$ is an isomorphism under the map $a \longrightarrow a \cap \mu_K$ where K contains the support of a and μ_K is from Lemma 1.11.

Theorem 1.18 generalizes Theorem 1.17 and one can prove 1.18 by first proving it when $K = pt$ and then by a patching argument (by Mayor-Vietoris).