1. Grassmann Bundles.

Note if you have a smooth embedded curve

$$\gamma : I \to \mathbb{R}^{k+1}, \quad I \subset \mathbb{R}$$

then we have a Gauss map $I \to S^k$ sending x to $\frac{d}{dt}\gamma/|\frac{d}{dt}\gamma|$. The image of this map only depends on I up to *oriented reparameterization*. If we wish to forget the orientation of the curve γ, then we have a Gauss map $I \to \mathbb{RP}^n$ sending x to $\pm\frac{d}{dt}\gamma/|\frac{d}{dt}\gamma|$. This Guass map (up to reparameterization) only depends on the submanifold $\gamma(I) \subset \mathbb{R}^{k+1}$ We wish to generalize this Gauss map from curves to general submanifolds of Euclidean space. In other words, if $M^n \subset \mathbb{R}^{n+k}$ then its tangent space $TM^n \subset T_x\mathbb{R}^{n+k} = \mathbb{R}^{n+k}$ is an n-dimensional subspace and hence we need a space which parameterizes all n dimensional subspaces of $\mathbb{R}^{n+k}$.

Definition 1.1. The **Grassmann manifold** $G_n(\mathbb{R}^{n+k})$ is the set of all n-dimensional subspaces of $\mathbb{R}^{n+k}$. This is given the following topology:

An n**-frame** is a collection of n linearly independent vectors in $\mathbb{R}^{n+k}$. The set of n-frames forms an open subset

$$V_n(\mathbb{R}^{n+k}) \subset (\mathbb{R}^{n+k})^n$$

called the **Stiefel-manifold.** We have a canonical surjective function

$$q : V_n(\mathbb{R}^{n+k}) \twoheadrightarrow G_n(\mathbb{R}^{n+k})$$

sending an n-frame to the space it spans. The topology of $G_n(\mathbb{R}^{n+k})$ is then the quotient topology (i.e. $U \subset G_n(\mathbb{R}^{n+k})$ is open if and only if $q^{-1}(U)$ is open).

Note that if $n = 1$ then $G_1(\mathbb{R}^{1+k}) = \mathbb{RP}^k$.

Lemma 1.2. $G_n(\mathbb{R}^{n+k})$ has the structure of a smooth manifold.

Proof. First of all, define $V_n^o(\mathbb{R}^{n+k}) \subset V_n(\mathbb{R}^{n+k})$ to be the set of orthonormal n-frames and define

$$q_o : V_n^o(\mathbb{R}^{n+k}) \twoheadrightarrow G_n(\mathbb{R}^{n+k}), \quad q_o = q|_{V_n^o(\mathbb{R}^{n+k})}.$$

Then the topology on $G_n(\mathbb{R}^{n+k})$ is the quotient topology of $V_n^o(\mathbb{R}^{n+k})$ induced by q_o. Since the fibers of q_o are compact, and since $V_n^o(\mathbb{R}^{n+k})$ is a metric space, we get that $G_n(\mathbb{R}^{n+k})$ is Hausdorff (in fact this implies it is a metric space). It is also paracompact since $V_n^o(\mathbb{R}^{n+k})$ is paracompact.

We now need to construct charts around each point V of $G_n(\mathbb{R}^{n+k})$. Here $V \subset \mathbb{R}^{n+k}$. Let $V^\perp$ be the set of vectors orthogonal to V. Hence we have an orthogonal projection $P : \mathbb{R}^{n+k} = V \oplus V^\perp \twoheadrightarrow V$ sending $v \oplus v^\perp$ to v. Let $U_V \subset G_n(\mathbb{R}^{n+k})$ be the set of $V' \in G_n(\mathbb{R}^{n+k})$ so that $P|_{V'}$ is an isomorphism. Then U_V is the set of graphs of linear maps $V \to V^\perp$. Hence we have a natural isomorphism $\Phi_V : U_V \cong Hom(V, V^\perp)$. Choose a basis for V and $V^\perp$. Since $dim(V) = n$ and $dim(V^\perp) = k$, we can choose an isomorphism $\Psi_V : Hom(V, V^\perp) \cong \mathbb{R}^{nk}$ sending a map to its corresponding $n \times k$ matrix with respect to the above basis. Hence $\Psi_V \circ \Phi_V : U_V \to \mathbb{R}^{nk}$ is our chart.

The transition maps are smooth for the following reason. The map $\Psi_{V'} \circ \Phi_{V'} \circ (\Psi_V \circ \Phi_V)^{-1}$ sends an $n \times k$ matrix to its graph Γ in $\mathbb{R}^{n+k}$ and then applies a fixed linear transformation T to this graph and then gives us the corresponding matrix of $T(\Gamma)$. Exercise: Such a map is smooth when defined. □

Definition 1.3. The **universal bundle** over $G_n(\mathbb{R}^{n+k})$ is the subset

$$\gamma_k^n \equiv \gamma^n(\mathbb{R}^{n+k}) \equiv \{(V, x) \in G_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k} \ : \ x \in V\} \subset G_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k}.$$

We also have a natural map: $\pi : \gamma^n(\mathbb{R}^{n+k}) \twoheadrightarrow G_n(\mathbb{R}^{n+k})$ sending (V, x) to V.

Lemma 1.4. The map π above makes $\gamma_k^n = \gamma^n(\mathbb{R}^{n+k})$ into a smooth vector bundle of rank k.

Proof. We just need to construct trivializations of π over each chart $\Psi_V \circ \Phi_V : U_V \to \mathbb{R}^{nk}$ from the previous proof. Here Ψ_V involved choosing a basis for V and $V^\perp$. Hence we have a natural isomorphism $\phi : V \to \mathbb{R}^n$. The trivialization τ sends $(B, x) \in U_V \times \mathbb{R}^{n+k}$ where $x \in B$ to $(\Psi_V(\Phi(B)), \phi(P(x)) \in \mathbb{R}^{nk} \times \mathbb{R}^k$.

Exercise: show that the transition maps are smooth bundle isomorphisms. □

Lemma 1.5. For any rank n vector bundle $\pi : E \to B$ over a compact normal base, there is a (large) $k \in \mathbb{N}$, a map $f : B \to G_n(\mathbb{R}^{n+k})$ so that E is isomorphic to $f^*(\gamma_k^n)$.

In other words, every rank n vector bundle is isomorphic to the pullback of the universal vector bundle with respect to some map. If the vector bundle is smooth then this can be done smoothly.

Proof. It is sufficient to construct a continuous map $f : E \to \mathbb{R}^k$ so that the restriction of f to each fiber $\pi^{-1}(b)$ is a linear map from $\pi^{-1}(b)$ to $\mathbb{R}^k$. This then gives us a map $F : E \to \gamma_k^n$ sending $x \in E$ to $(\pi(\pi^{-1}(x)), f(x)) \in \gamma_k^n$ which is a fiberwise isomorphism which would prove the lemma.

We construct f chart by chart. Choose finite open covers $(U_i)_{i\in S}$, $(V_i)_{i\in S}$ and $(W_i)_{i\in S}$ so that $\overline{W}_i \subset V_i$ and $\overline{V}_i \subset U_i$ and so that $E|_{U_i}$ is trivial for each $i \in S$. Choose continuous functions $\lambda_i : U_i \to \mathbb{R}$ equal to 0 outside V_i and equal to 1 along W_i (this can be done using the normal property). We have trivializations $\tau_i : E|_{U_i} \to U_i \to \mathbb{R}^n$. Hence we have fiberwise linear maps $h_i \equiv pr_2 \circ \tau_i : E_{U_i} \to \mathbb{R}^n$ where $pr_2 : U_i \times \mathbb{R}^n \to \mathbb{R}^n$ is the natural projection map.

Define:

$$h_i : E \to \mathbb{R}^n, \quad h_i(x) \equiv \begin{cases} 0 & \text{if } \pi(x) \notin U_i \\ \lambda_i(\pi(x)).h_i(x) & \text{if } \pi(x) \in U_i \end{cases}$$

Then the map

$$f : E \to \oplus_{i\in S}\mathbb{R}^n \cong \mathbb{R}^{n|S|}, \quad f((x_i)_{i\in S}) = \oplus_{i\in S} h_i(x_i)$$

is a map whose restriction to each fiber is a linear embedding.

Note that if $\pi : E \to B$ is a smooth vector bundle then all of the above maps can be chosen to be smooth. □

Definition 1.6. The inclusion map $\mathbb{R}^{k'} \subset \mathbb{R}^k, \quad x \to (x, 0)$ for $k' \leq k$ gives us a natural inclusion map $G_n(\mathbb{R}^{n+k'}) \hookrightarrow G_n(\mathbb{R}^{n+k})$ and also an inclusion homomorphism of bundles $\gamma_{k'}^n \hookrightarrow \gamma_k^n$. Here $\gamma_{k'}^n$ is canonically isomorphic to the pullback of γ_k^n.

We define $G_n(\mathbb{R}^\infty)$ to be the direct limit $\lim_{k\to\infty}(G_n(\mathbb{R}^{n+k}))$ with the direct limit topology. In other words $U \subset G_n(\mathbb{R}^\infty)$ is open if and only if its restriction to $G_n(\mathbb{R}^{n+k})$ is open for all $k \in \mathbb{N}$.

Similarly we can define γ_∞^n to be the direct limit of γ_k^n with the direct limit topology.

Note that we can describe $G_n(\mathbb{R}^\infty)$ as follows: We define $\mathbb{R}^\infty = \lim_k \mathbb{R}^k$ with the direct limit topology. This is the vector space spanned by finite sequences $(x_1, \cdots, x_k)$. Hence we can define $G_n(\mathbb{R}^\infty)$ in the usual way (as a topological vector bundle) with respect to n-dimensional subspaces of $\mathbb{R}^\infty$.

Lemma 1.7. We have that γ_∞^n is a topological vector bundle of $G_n(\mathbb{R}^\infty)$.

Proof. Let $V \in G_n(\mathbb{R}^\infty)$. Let U_V be the open set given by the union of the open sets $U_V^k \subset G_n(\mathbb{R}^{n+k})$ where U_V^k is the set of subspaces of $\mathbb{R}^{n+k}$ which are transverse to $V^\perp$ as before. Then we have trivializations $\tau_k : \gamma_k^n|_{U_V^k} \to U_V^k \times \mathbb{R}^k$ along U_i as before. The trivialization $\tau : \gamma_\infty^n|_{U_V} \to U_V \times \mathbb{R}^k$ is the direct limit of these trivializations (I.e it is the unique map whose restriction to U_V^k is τ_i). □

Theorem 1.8. Any bundle $\pi : E \to B$ over a paracompact base B of rank n admits a map $f : B \to G_n(\mathbb{R}^\infty)$ so that E is isomorphic to $f^*\gamma_\infty^n$.

Definition 1.9. Let $h, h' : E \to \gamma_\infty^n$ be bundle maps. A **smooth family of bundle maps** $(h_t : E \to \gamma_\infty^n)_{t\in[0,]}$ **joining** h **with** h' is a bundle map $\widetilde{h} : [0,1] \times E \to \gamma_\infty^n$ so that $\widetilde{h}|_{\{t\}\times E=E} = h_t$ and so that $h_0 = h$ and $h_1 = h'$. We say that h, h' are **bundle homotopic** there is a smooth family of bundle maps joining h with h'.

Theorem 1.10. Any two bundle maps $h, h' : E \to \gamma_\infty^n$ which are an isomorphism on each fiber are bundle homotopic.

We will not prove this. The proof is contained in Milnor's book (Chapter 5) and is accessible. The idea of the proof of Theorem 1.8 is to use the fact that a paracompact space is an increasing union of compact spaces and then use Lemma 1.5 some sort of limit argument.

Instead of proving these theorems we will talk about **classifying spaces**. This is a generalization of $\gamma_\infty^1 \to G_n(\mathbb{R}^\infty)$.