1. Existence and Uniqueness of Stiefel Whitney Classes .

We will now prove the following theorem (which was stated earlier)

Theorem 1.1. To each topological vector bundle $\pi : E \longrightarrow B$ of rank n, there is a sequence of cohomology classes

$$w_i(E) \in H^i(B; \mathbb{Z}/2), \quad i = 0, 1, 2, \cdots$$

where $w_i(E)$ is called the **ith Stiefel-Whitney class** so that:

(Stiefel-1) **rank axiom** $w_0(E) = 1$ and $w_i(E) = 0$ for $i > n$.

(Stiefel-2) **Naturality**: For any continuous map $f : B' \longrightarrow B$, we have that $w_i(f^*E) = f^*(w_i(E))$. Also isomorphic vector bundles have the same Stiefel-Whitney classes.

(Stiefel-3) **The Whitney Product Theorem**: Let $\pi : E \longrightarrow B$, $\pi' : E' \longrightarrow B$ be fiber bundles over the same base B. Then

$$w_k(E \oplus E') = \sum_{i=0}^{k} w_i(E) \cup w_{k-i}(E').$$

(Stiefel-4) **normalization axiom**: $w_1(\mathcal{O}_{\mathbb{RP}^1}(-1)) \neq 0$ where $\mathcal{O}_{\mathbb{RP}^1}(-1)$ is the natural vector bundle on $\mathbb{RP}^1$ introduced earlier.

Proof. Existence: For each vector bundle $\pi : E \longrightarrow B$, let $F_E : B \longrightarrow Gr_n(\mathbb{R}^\infty)$ be the classifying map. Let $h_n : (\mathbb{RP}^\infty)^n \longrightarrow Gr_n(\mathbb{R}^\infty)$ be the classifying map for the bundle $\oplus_{i=1}^n p_i^* \gamma_\infty^1$ where $p_i : (\mathbb{RP}^\infty)^n \longrightarrow \mathbb{RP}^\infty$ is the ith projection map. Recall that $H^*((\mathbb{RP}^\infty)^n) = (\mathbb{Z}/2\mathbb{Z})[a_1, \cdots, a_n]$ and

$$h_n^* : H^*(Gr_n(\mathbb{R}^\infty); \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^*((\mathbb{RP}^\infty)^n)$$

is injective with image equal to

$$(\mathbb{Z}/2\mathbb{Z})[\sigma_1, \cdots, \sigma_n]$$

where σ_i is the ith symmetric polynomial in $a_1, \cdots, a_n$. We define $w_i(E) \equiv F_E^*(\sigma_i)$ for every vector bundle E. We need to show that $w_i(E)$ satisfies the axioms (Stiefel-1) - (Stiefel-4).

Rank Axiom: This is satisfied by definition.

Naturality: For any vector bundle $\pi : E \longrightarrow B$ and any map $f : B' \longrightarrow B$, we have that $F_{f^*E} = F_E \circ f$. Hence $w_i(f^*E) = f^* F_E^* \sigma_i = f^* w_i(E)$. Also if $\pi' : E' \longrightarrow B$ is isomorphic to $\pi : E \longrightarrow B$ then f_E is homotopic to $f_{E'}$ which means that $w_i(E) = f_E^*(\sigma_i) = f_{E'}^*(\sigma_i) = w_i(E')$.

The Whitney Product Theorem: Define $G \equiv Gr_n(\mathbb{R}^\infty)$, $G' \equiv Gr_{n'}(\mathbb{R}^\infty)$, $G'' \equiv Gr_{n+n'}(\mathbb{R}^\infty)$, $\gamma \equiv \gamma_\infty^n$, $\gamma' \equiv \gamma_\infty^{n'}$ and $\gamma_3 \equiv \gamma_\infty^{n+n'}$. Let

$$p : G \times G' \longrightarrow G, \quad p' : G \times G' \longrightarrow G'$$

be the natural projection map. Let $F : G \times G' \longrightarrow G_3$ be the classifying map for $p^*\gamma \oplus {p'}^*\gamma'$. We have the following commutative diagram (up to homotopy):

$$\begin{array}{ccc} (\mathbb{RP}^\infty)^n \times (\mathbb{RP}^\infty)^{n'} & \xrightarrow{\widetilde{F}} & (\mathbb{RP}^\infty)^{n+n'} \\ \Big\downarrow h_n \times h_{n'} & & \Big\downarrow h_{n+n'} \\ G \times G' & \xrightarrow{F} & G_3 \end{array}$$

The top horizontal map $\widetilde{F}$ is the natural homeomorphism given by including $(\mathbb{RP}^\infty)^n$ into the first n factors of $(\mathbb{RP}^\infty)^{n+n'}$ and by including including $(\mathbb{RP}^\infty)^{n'}$ into the last n' factors of $(\mathbb{RP}^\infty)^{n+n'}$. We have

$$H^*((\mathbb{RP}^\infty)^n;\mathbb{Z}/2/Z) = (\mathbb{Z}/2\mathbb{Z})[a,\cdots,a_n], \quad H^*((\mathbb{RP}^\infty)^{n'};\mathbb{Z}/2/Z) = (\mathbb{Z}/2\mathbb{Z})[a'_1,\cdots,a'_{n'}]$$

and

$$H^*((\mathbb{RP}^\infty)^{n_3};\mathbb{Z}/2/Z) = (\mathbb{Z}/2\mathbb{Z})[a''_1,\cdots,a''_{n+n'}].$$

Also

$$H^*(G;\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[\sigma_1,\cdots,\sigma_n], \quad H^*(G';\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[\sigma'_1,\cdots,\sigma'_{n'}]$$

and

$$H^*(G_3;\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})[\sigma''_1,\cdots,\sigma''_{n+n'}]$$

where σ_i is the ith symmetric polynomial in a_i and σ'_i is the ith symmetric polynomial in a'_i and σ''_i is the ith symmetric polynomial in a''_i.

Then $\widetilde{F}^*(c_i) = a_i$ for $i \leq n$ and $\widetilde{F}^*(c_i) = a'_{i-n}$ for $i > n$. Since the maps

$$h_n^* : H^*(G;\mathbb{Z}/2\mathbb{Z}) \longrightarrow H^*((\mathbb{RP}^\infty)^n;\mathbb{Z}/2)$$
$$h_{n'}^* : H^*(G';\mathbb{Z}/2\mathbb{Z}) \longrightarrow H^*((\mathbb{RP}^\infty)^{n'};\mathbb{Z}/2)$$
$$h_{n+n'}^* : H^*(G_3;\mathbb{Z}/2\mathbb{Z}) \longrightarrow H^*((\mathbb{RP}^\infty)^{n+n'};\mathbb{Z}/2)$$

are injective with image given by the symmetric polynomials in a_i, a'_i and a''_i respectively, we get that

$$F^*(\sigma'_i) = \sum_{j=1}^{i} \sigma_j \cup \sigma_{i-j}. \tag{1}$$

Now let $\pi : E \longrightarrow B$, $\pi' : E' \longrightarrow B$ be fiber bundles over the same base B of rank n and n' respectively. Then we have classifying maps F_E, $F_{E'}$ and $F_{E\oplus E'}$. Let $F_E \times F_{E'} : B \longrightarrow G_1 \times G_2$. Then $F \circ (F_E \times F_{E'})$ is homotopic to $F_{E\oplus E'}$ since G_3 is a classifying space. Combining this with formula (1) tells us that

$$w_i(E \oplus E') = F^*_{E\oplus E'}(\sigma''_i) = (F_E \times F_{E'})^*(\sum_{j=1} \sigma_j \cup \sigma'_{i-j}) = \sum_{j=1}^{i} w_i(E) \cup w_i(E').$$

normalization axiom: This follows from the fact that the identity map $\mathbb{RP}^1 \longrightarrow \mathbb{RP}^1$ is the classifying map for γ_1^1.

Uniqueness of Stiefel-Whitney Classes: Suppose we have Stiefel-Whitney classes $w_i(E)$ satisfying (Stiefel-1) - (Stiefel-4). Since every rank n bundle is the pullback of γ_∞^n, we only need to calculate $w_i(\gamma_\infty^n) \in H^i(Gr_n(\mathbb{R}^\infty))$. Also since the natural map

$$h_n^* : H^*(Gr_n(\mathbb{R}^\infty);\mathbb{Z}/2\mathbb{Z}) \longrightarrow H^*((\mathbb{RP}^\infty)^n)$$

is injective, and since $h_n^*(\gamma_\infty^n) = \oplus_{i=1}^n p_i^*\gamma_\infty^1$, it is in fact sufficient for us to calculate $w_i(\oplus_{i=1}^n p_i^*\gamma_\infty^1)$. By the Whitney sum formula and naturality applied to p_i, it is sufficient for us to calculate $w_i(\gamma_\infty^1) \in H^i(\mathbb{RP}^\infty)$ for all i. By the rank axiom $w_i(\gamma_\infty^1) = 0$ for $i > 1$ and $w_0(\gamma_\infty^1) = 1$. Also since the map $H^1(\mathbb{RP}^\infty) \leq H^1(\mathbb{RP}^1;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ is an isomorphism and $\gamma_\infty^1|_{\mathbb{RP}^1} = \gamma_1^1$, we have by the normalization axiom that $w_1(\gamma_\infty^1) = 1$. Hence these classes are unique. □

Corollary 1.2. We have that $w_n(E) = e(E;\mathbb{Z}/2)$.

Proof. This follows from naturality of the unoriented Euler class and $w_n(E)$ and from the fact that $e(\gamma_\infty^n;\mathbb{Z}/2) = \sigma_n = w_n(\gamma_\infty^n;\mathbb{Z}/2)$. □