1. Oriented Euler Class.

The problem with Stiefel-Whitney classes is that they are only classes in cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients. Sometimes more information can be obtained if we have classes with $\mathbb{Z}$ coefficients.

Definition 1.1. A vector bundle is **orientable** if its structure group can be reduced to invertible matrices with positive determinant $GL^+(n, \mathbb{R}) \subset GL(n, \mathbb{R})$. Equivalently, a vector bundle $\pi : E \longrightarrow B$ of rank n is **orientable** if its highest wedge power $\wedge^n E$ is a trivial line bundle. An **orientation** is a choice of trivialization of this line bundle up to homotopy I.e. a choice of isomorphism $\wedge^n E \cong B \times \mathbb{R}$ up to homotopy.

A vector bundle $\pi : E \longrightarrow B$ is **oriented** if it has a fixed choice of orientation $\tau : \wedge^n E \longrightarrow B \times \mathbb{R}$.

If B is connected then there are only two choices of isomorphism up to homotopy since automorphisms of $B \times \mathbb{R}$ correspond to functions $f : B \longrightarrow \mathbb{R} - 0$ where f corresponds to the automorphism

$$B \times \mathbb{R} \longrightarrow B \times \mathbb{R}, \quad (b, t) \longrightarrow (b, f(b)t).$$

We can also define an orientation in the following way: Recall that $H_n(V, V - 0; \mathbb{Z}) = \mathbb{Z}$ for any vector space V of dimension n.

Definition 1.2. A **homological orientation** on a rank n vector bundle $\pi : E \longrightarrow B$ consists of class $\mu_x \in H_n(E|_x, E|_x - 0; \mathbb{Z})$ for each $x \in B$ so that for each $x \in B$ there is a neighborhood $N_x \ni x$ of x in B and a class $\mu_{N_x} \in H_n(E|_{N_x}, E|_{N_x} - N_x; \mathbb{Z})$ so that the restriction of μ_{N_x} to $H_n(E|_y, E|_y - 0; \mathbb{Z})$ is μ_y for all $y \in B$.

Lemma 1.3. There is a $1 - 1$ correspondence between homological orientations and orientations.

As a result we will just use the word **orientation** instead of **homological orientation**.

Proof. We will use the following facts:

Fact 1: Let $\mu \in H_n(\mathbb{R}^n, \mathbb{R} - 0; \mathbb{Z})$ be a generator of this rank 1 free abelian group. If A is a linear map with positive determinant then $A_*(\mu) = \mu$.

Fact 2: If V is a vector space with a choice of linear isomorphism $\wedge^n V \longrightarrow \mathbb{R} = \wedge^n \mathbb{R}^n$. Then any two linear maps $B_1, B_2 : V \longrightarrow \mathbb{R}^n$ inducing the above isomorphism up to homotopy give the same isomorphism $H_n(V, V - 0; \mathbb{Z}) \longrightarrow H_n(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z})$.

Therefore fact 1 and fact 2 tells us that a choice of trivialization $\wedge^n V \longrightarrow \mathbb{R}$ give us a canonical choice of generator $\mu \in H_n(V, V - 0; \mathbb{Z})$.

Suppose that $\pi : E \longrightarrow B$ is oriented. Then we have a a choice of trivialization $\tau : \wedge^n E \longrightarrow B \times \mathbb{R}$. The above discussion tells us that τ induces a canonical choice of generator. $\mu_x \in H_n(E|_x; E|_x - 0; \mathbb{Z})$ Now we define N_x to be a neighborhood of x so that $E|_{N_x} = N_x \times \mathbb{R}^n$. Then $\mu_N \in H_n(N_x \times \mathbb{R}^n, N_x \times (\mathbb{R}^n - 0)); \mathbb{Z}) = \mathbb{Z}$ is the unique class restricting to μ_x. This restricts to μ_y for each $y \in N_x$ be Fact 1 and Fact 2.

Conversely if $\pi : E \longrightarrow B$ is homologically oriented. Then the choice of μ_x give us local sections σ_{N_x} on small neighborhoods N_x of B with the property that σ_{N_x} and σ_{N_y} are positive multiples of each other. Hence we can patch these sections together to get a nowhere zero section of $\wedge^n E$. Hence $\pi : E \longrightarrow B$ is orientable. □

We also have an oriented version of the Thom Isomorphism theorem:

Theorem 1.4. (Thom Isomorphism Theorem) Let $\pi : E \longrightarrow B$ be an oriented vector bundle. Then there is a class $\widetilde{e}(E) \in H^*(E, E - B; \mathbb{Z})$ so that the map

$$H^*(B;\mathbb{Z}) \longrightarrow H^*(E;\mathbb{Z}), \quad \alpha \longrightarrow \pi^*\alpha \cup \widetilde{e}(E)$$

is an isomorphism.

Corollary 1.5. If $\pi : E \longrightarrow B$ is oriented then there is a class $\mu_B \in H_n(E; E - B; \mathbb{Z})$ whose restriction to $H_n(E|_x, E|_x - 0; \mathbb{Z})$ is μ_x for each $x \in B$.

This theorem follows from the following Theorem:

Theorem 1.6. Let $\pi : E \longrightarrow B$ be an oriented vector bundle. Choose an identification of the fiber F with $\mathbb{R}^n$ so that its orientation is preserved (i.e. the map $\wedge^n\mathbb{R}^n = \mathbb{R} = \wedge^n F$ corresponds to our orientation).

Then there is a class $\widetilde{e}(E) \in H.(E, E-B; \mathbb{Z})$ so that the restriction of $\widetilde{e}(E)$ to $H^*(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z}) \cong \mathbb{Z}$ is 1 where 1 corresponds to the boundary of the linear $n+1$-simplex in $\mathbb{R}^n$ with vertices at the basis vectors $e_1, \cdots, e_n$ and at $(-1, \cdots, -1)$.

Also this class is unique by the Thom isomorphism theorem.

Definition 1.7. The class $\widetilde{e}(E)$ is called the **fundamental class of** E.

The Thom Isomorphism Theorem follows from the above theorem combined with the Leray-Hirsch theorem. But..... The Leray-Hirsch theorem only works when one has coeffients over a field F. Hence the map

$$H^*(B;\mathbb{K}) \longrightarrow H^*(E;\mathbb{K}), \quad \alpha \longrightarrow \pi^*\alpha \cup \widetilde{e}(E)$$

is an isomorphism over any field $\mathbb{K}$. But this is enough to show that

$$H^*(B;\mathbb{Z}) \longrightarrow H^*(E;\mathbb{Z}), \quad \alpha \longrightarrow \pi^*\alpha \cup \widetilde{e}(E)$$

is an isomorphism.

Sketch of the proof of Theorem 1.6. The proof of this theorem first uses Mayor-Vietoris and the five lemma to show that there is a class $\widetilde{e}(E)$ whose restriction to any fiber $H^*(F, F_0; \mathbb{Z}) = H^*(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z}) \cong \mathbb{Z}$ is 1. This is where orientability is needed. The reason is that there are two possible isomorphisms $H^*(F, F_0; \mathbb{Z}) \cong \mathbb{Z}$ where one is minus one times the other. The choice of orientation fixes such an isomorphism since it identifies $\wedge^n F$ with $\wedge^n\mathbb{R}$ in a canonical way.

We will not give a detailed proof of the Thom isomorphism here as it is essentially the same as the non-oriented Thom Isomorphism theorem.

Definition 1.8. The **Euler Class** of a rank n oriented vector bundle $\pi : E \longrightarrow B$ is the class $e(E) \in H^n(B; \mathbb{Z})$ given by the image of $\widetilde{e}(E)$ under the map

$$H^n(E; E_0; \mathbb{Z}) \longrightarrow H^n(E; \mathbb{Z}) \longrightarrow H^n(B; \mathbb{Z})$$

where the last morphism is the zero section inclusion map.

Note that the image of $e(E)$ inside $H^*(B; \mathbb{Z}/2\mathbb{Z})$ is the unoriented Euler class $e(E; \mathbb{Z}/2\mathbb{Z})$.

Lemma 1.9. $e(B \times \mathbb{R}^n) = 0$.

Exercise.

We have the following corollary of Theorem 1.6:

Corollary 1.10. There is a $1-1$ correspondence between orientations on E and classes $\widetilde{e}(E)$ satisfying the properties of Theorem 1.6.

To prove this corollary one needs to show that $\widetilde{e}(E)$ is unique. This follows from the Thom isomorphism theorem.

Theorem 1.11. Let $\pi : E \longrightarrow B$, $\pi : E' \longrightarrow B$ be vector bundles. Then $e(E \oplus E') = e(E) \cup e(E')$.

Proof. Consider the bundle

$$E \times E' \longrightarrow B \times B, \quad (e, e') \longrightarrow (\pi(e), \pi'(e')).$$

Let $P : E \times E' \longrightarrow E$, $P' : E \times E' \longrightarrow E'$ $p : B \times B' \longrightarrow B$, $p' : B \times B' \longrightarrow B'$ be the natural projection maps.

Then $\widetilde{e}(E \times E') = P^*\widetilde{e}(E) \cup P'^*\widetilde{e}(E')$. Hence $e(E \times E') = p^*e(E) \cup P'^*e(E')$.

Let

$$\Delta : B \longrightarrow B \times B, \quad \Delta(x) = (x, x)$$

be the diagonal inclusion map. Then since $E \oplus E' \cong \Delta^*(E \times E')$, we have that $e(E \oplus E') = \Delta^*(p^*e(E) \cup P'^*e(E')) = e(E) \cup e(E')$. □

Corollary 1.12. Suppose that the vector bundle $\pi : E \longrightarrow B$ admits a section $s : B \longrightarrow E$ so that $s(x) \neq 0$ for all $x \in B$. Then $e(E) = 0$.

Proof. Let $L \subset E$ be the rank 1 vector subbundle whose fiber at $x \in B$ consists of the set of vectors of the form $cs(x)$ where $c \in \mathbb{R}$. The L admits a section s and hence L is trivial. Therefore $E \cong \mathbb{R} \oplus (E/L)$. Hence $e(E) = e(\mathbb{R}) \cup e(E/L) = 0 \cup e(E/L) = 0$. □