MAT 322 Spring 2018 Midterm I Exam

Name: **ID Number:**

Problem	1	2	3	4	5	Total
Points	20	20	20	20	20	100
Score						

Directions: Do all of your work on these exam sheets; you may use the backs of pages as needed.

Show all your relevant work: Partial credit will not be given without justification or reasoning of your solutions.

1. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be connected open domains in $\mathbb{R}^n$, $\mathbb{R}^m$ respectively and suppose $F : U \to V$ is a diffeomorphism, i.e. F is a C^1 mapping onto V with C^1 inverse $F^{-1} : V \to U$. Prove that $m = n$. (This is called invariance of domain for C^1 mappings).

2. Suppose $f = f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is C^1 and $\partial_y f = 0$ everywhere. Prove that $f = f(x, y)$ is independent of y. If $\partial_x f = \partial_y f = 0$ everywhere on $\mathbb{R}^2$, prove that f is constant.

(Hint: use the mean value theorem).

Let $A = \{(x, y) \in \mathbb{R}^2 : x < 0 \text{ or } x \geq 0 \text{ and } y \neq 0\}$. If $f : A \to \mathbb{R}$ satisfies $Df = 0$ prove that f is constant.

Find a C^1 function $f : A \to \mathbb{R}$ such that $\partial_y f = 0$ but f is not independent of y.

3. Let Sym_n be the subspace of symmetric matrices in the space $M_n(\mathbb{R})$ of $n \times n$ matrices. This is a linear subspace isomorphic to $\mathbb{R}^{n(n+1)/2} \subset \mathbb{R}^{n^2}$. The determinant function

$$det : Sym \to \mathbb{R},$$

is a C^1 function. Compute the derivative $D_I det$ of the determinant at the identity matrix I.

(You may use the fact that any symmetric matrix is diagonalizable, with eigenvalues on the diagonal).

Extra Credit (10pts). Compute the derivative $D_I det$, for

$$det : M_n(\mathbb{R}) \to \mathbb{R},$$

4. Let Q be a rectangle in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function.
(a) State a necessary and sufficient condition for f to be integrable on Q.

(b) Suppose $f = 0$ outside a closed set Z of measure zero in Q. Prove that f is integrable and

$$\int_Q f = 0.$$

(c) Show by an example that (b) is false if Z is not closed.

5. Let $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and suppose $f : Q \to \mathbb{R}$ is continuous. Define $F : Q \to \mathbb{R}$ by

$$F(x) = \int_{[a_1,x_1]\times\cdots\times[a_n,x_n]} f.$$

Determine the partial derivatives $\partial_i f(x)$, for x in the interior of Q.