MAT 322 Final Exam
Spring 2021

May 5, 2021

Question	Points possible	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	60	

Instructions

1. You may use your textbook, course notes and homework.
2. Write the solution to each problem neatly on a separate piece of paper or your virtual notebook. You don't have to copy down the problem statement.

Declaration

Read and sign on your own paper (you do not need to copy the full statement):

1. I understand that I am not allowed to use the internet, computer algebra systems, other people, or any outside resource to take this test.
2. I understand that breaking any of these rules will result in a grade of F for this course, and I will be reported to the SBU Academic Judiciary.

Date

Signature

Exam

In problems 1 and 2, points in $\mathbb{R}^3$ are denoted in coordinates by (x, y, z).

1. Let
$$\omega = (x^2 + y^2)\,dx \wedge dy + e^x\,dx \wedge dz + e^y\,dy \wedge dz.$$

(a) Find a 1-form θ on $\mathbb{R}^3$ satisfying $d\theta = \omega$.

(b) Calculate $\displaystyle\int_H \omega$, where H is the 2-manifold $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^6 = 1, z \geq 0\}$ oriented with upward-pointing unit normal vector.

2. Consider the 2-form ω on $\mathbb{R}^3 \setminus \{0\}$ given by
$$\omega = r^{-3}(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy),$$
where $r = \sqrt{x^2 + y^2 + z^2}$.

(a) Show that ω is closed.

(b) For all $r > 0$, let $S_r = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = r^2\}$ be the sphere of radius r, oriented with outward pointing unit normal vector. Show (by direct computation) that
$$\int_{S_r} \omega = 4\pi = v(S_1)$$
for all $r > 0$.

(c) Determine whether or not ω is an exact 2-form.

3. Let $O(n)$ denote the group of orthogonal $n \times n$ matrices. Recall that an $n \times n$ matrix A is orthogonal if $AA^T = I$, where I is the $n \times n$ orthogonal matrix.

(a) Show that $O(n)$ is a compact manifold in $\mathbb{R}^{n^2}$ without boundary. [You may use the result in Exercise 2 on p. 208 of the textbook.]

(b) What is the dimension of $O(n)$?

(c) Identify the tangent space $T_I(O(n))$ of $O(n)$ at the identity matrix I. That is, what space of matrices does it correspond to?

4. Let $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}$ denote the standard basis for $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, and $\phi_1, \ldots, \phi_n, \chi_1, \ldots, \chi_n$ the corresponding elementary 1-tensors. For each alternating k-tensor g on $\mathbb{R}^{2n}$, define inductively $g^{\wedge 1} = g$ and $g^{\wedge(i+1)} = g \wedge g^{\wedge i}$ for each $i > 1$. Let
$$\omega = \phi_1 \wedge \chi_1 + \cdots + \phi_n \wedge \chi_n.$$
Determine $\omega^{\wedge n}$.

5. Let $e_1, e_2, \ldots, e_n$ denote the standard basis for $\mathbb{R}^n$, and let $\phi_1, \phi_2, \ldots, \phi_n$ denote the corresponding elementary 1-tensors. Define the *elementary symmetric k-tensors* by

$$\phi_{i_1} \vee \phi_{i_2} \vee \cdots \vee \phi_{i_k} = \sum_{\sigma \in S_k} (\phi_{i_{\sigma(1)}} \otimes \phi_{i_{\sigma(2)}} \otimes \cdots \otimes \phi_{i_{\sigma(k)}}). \tag{1}$$

Here, $(i_1, \ldots, i_k)$ is an arbitrary k-tuple taking values in $\{1, \ldots, n\}$.

(a) Prove that the tensor defined in (1) is indeed symmetric.

(b) Prove that

$$\{\phi_{i_1} \vee \cdots \vee \phi_{i_k} : 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n\}$$

is a basis for the space of symmetric k-tensors on $\mathbb{R}^n$.

(c) Determine the dimension $d(n,k)$ of the space of symmetric k-tensors on $\mathbb{R}^n$. [Hint: the answer is an expression involving the binomial coefficient $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for non-negative integers a, b, though the exact expression is probably not obvious in advance. The main step is to find a recursive formula for $d(n,k)$. The base case $d(n,0)$ should be obvious, and it might help to look at the cases $d(n,1)$ and $d(n,2)$ as well to get a feel for the problem. The only preliminary fact that should be needed is *Pascal's triangle identity* $\binom{a}{b} + \binom{a}{b-1} = \binom{a+1}{b}$.]

6. Let M be a compact oriented $(k+l+1)$-manifold without boundary in $\mathbb{R}^n$. Let ω be a k-form and η be an l-form, both defined on an open set in $\mathbb{R}^n$ containing M. Prove that

$$\int_M \omega \wedge d\eta = a \int_M d\omega \wedge \eta$$

for some a, and determine a.