MAT 322 Final Exam
Spring 2022

May 4, 2022

Question	Points possible	Score
1	10	
2	7	
3	7	
4	10	
5	10	
6	16	
Total	60	

Instructions

1. You may use your textbook, course notes and homework.
2. Write the solution to each problem neatly on a separate piece of paper or your virtual notebook. You don't have to copy down the problem statement.

Declaration

Read and sign on your own paper (you do not need to copy the full statement):

1. I understand that I am not allowed to use the internet, computer algebra systems, other people, or any outside resource to take this test.
2. I understand that breaking any of these rules will result in a grade of F for this course, and I will be reported to the SBU Academic Judiciary.

________________ Date

____________________________ Signature

Exam

1. Let

$$S = \{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2, z \leq 2\}$$

and $\omega = (y - z)\, dx \wedge dy - y\, dx \wedge dz + y\, dy \wedge dz$. Orient S using the upward-pointing unit normal vector.

(a) (4 points) Evaluate $\displaystyle\int_S \omega$ directly as a surface integral.

(b) (3 points) Find a 1-form η such that $d\eta = \omega$.

(c) (3 points) Evaluate $\displaystyle\int_S \omega$ using Stokes' theorem.

2. (7 points) Let $S^3 = \{x \in \mathbb{R}^4 : \|x\| = 1\}$ be the unit sphere in $\mathbb{R}^4$. Evaluate

$$\int_{S^3} x_1\, dx_2 \wedge dx_3 \wedge dx_4 - x_2\, dx_1 \wedge dx_3 \wedge dx_4 + x_3\, dx_1 \wedge dx_2 \wedge dx_4 - x_4\, dx_1 \wedge dx_2 \wedge dx_3.$$

3. Define the function $F\colon \mathbb{R}^4 \to \mathbb{R}^2$ by

$$F(x, y, z, w) = (f(x, y, z) + w^2, x^2y + z^2 - w^3),$$

where $f\colon \mathbb{R}^3 \to \mathbb{R}$ is some smooth function satisfying $f(1, 1, 1) = -1$. Let $a = (1, 1, 1, 1) \in \mathbb{R}^4$.

(a) (4 points) Give the most general condition on the derivative Df that guarantees that there is a neighborhood A of a in the set $F^{-1}(0, 0) \subset \mathbb{R}^4$, a neighborhood $B \subset \mathbb{R}^2$ and a function $g\colon B \to \mathbb{R}^2$ such that the map $G\colon B \to A$ defined by $G(z, w) = (g(z, w), z, w)$ is a parametrization of A.

(b) (3 points) Compute $Dg(1, 1)$ assuming that $Df(1, 1, 1) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$.

4. (10 points) Let $f\colon \mathbb{R}^{n+k} \to \mathbb{R}^n$ be a function of class C^1, written in coordinates as $f = (f_1, \ldots, f_n)$. Let

$$N = \{x \in \mathbb{R}^{n+k} : f_1(x) = \cdots = f_{n-1}(x) = 0, f_n(x) \geq 0\}$$

and $M = \{x \in \mathbb{R}^{n+k} : f(x) = \mathbf{0}\}$. Assume that $Df(x)$ has rank n for all $x \in M$ and $D(f_1, \ldots, f_{n-1})(x)$ has rank $n - 1$ for all $x \in N$. Prove that N is a $(k + 1)$-dimensional manifold, M is a k-dimensional manifold, and $\partial N = M$. (See p. 208 in the textbook.)

5. Recall that $\mathcal{A}^k(\mathbb{R}^n)$ is the space of alternating k-tensors on $\mathbb{R}^n$. Let $v \in \mathbb{R}^n$. For all $k \geq 1$, define the map $\iota_v\colon \mathcal{A}^k(\mathbb{R}^n) \to \mathcal{A}^{k-1}(\mathbb{R}^n)$ (*interior multiplication by* v) by the formula

$$\iota_v(\omega)(x_1, \ldots, x_{n-1}) = \omega(v, x_1, \ldots, x_{n-1}).$$

(a) (2 points) Prove that $\iota_v \circ \iota_v = 0$ for all $v \in \mathbb{R}^n$.

(b) (8 points) Let $\omega \in \mathcal{A}^k(\mathbb{R}^n)$, $\eta \in \mathcal{A}^l(\mathbb{R}^n)$ and $v \in \mathbb{R}^n$. Show that

$$\iota_v(\omega \wedge \eta) = (\iota_v(\omega)) \wedge \eta + (-1)^k \omega \wedge (\iota_v(\eta)).$$

6. Let V be a finite-dimensional vector space. We say that an alternating 2-tensor $\omega \in \mathcal{A}^2(V)$ is *symplectic* (or *non-degenerate*) if for all non-zero $x \in V$ there exists $y \in V$ such that $\omega(x, y) \neq 0$.

(a) (3 points) Explain why there is no symplectic tensor on $\mathbb{R}^1$. Show that, in contrast, every non-zero alternating tensor $\omega \in \mathcal{A}^2(\mathbb{R}^2)$ is symplectic.

(b) (6 points) Prove that there is no symplectic tensor on $\mathbb{R}^n$ whenever n is odd.

[Hint: Assume a symplectic tensor exists. Construct inductively a sequence of 2-dimensional subspaces $S_1, S_2, \ldots, S_k$ for all $k \leq n/2$ such that that each restriction $\omega|_{S_i \times S_i}$ is symplectic, and such that $\omega(x, y) = 0$ whenever $x \in S_i$, $y \in S_j$ for some $i \neq j$. If n is odd, we have an extra dimension left at the end. Derive a contradiction in this case.]

(c) (4 points) Define the k-th wedge product ω^k inductively by $\omega^1 = \omega$ and $\omega^k = \omega^{k-1} \wedge \omega$ for $k \geq 2$. Show that if a 2-tensor $\omega \in \mathcal{A}^2(\mathbb{R}^{2k})$ is not symplectic, then $\omega^k = 0$.

(d) (3 points) Give an example of a symplectic tensor on $\mathbb{R}^4$.

[Hint: based on part (c), find a 2-tensor for which $\omega \wedge \omega \neq 0$.]