MAT 322 MIDTERM 2 REVIEW

The test will consist of six short answer/computational problems (worth 25 points) and one proof (15 points). The short answer problems will focus on definitions and statements of theorems, computations, relevant examples, homework problems, and the relationship between the various concepts. You may be asked to provide a short argument justifying your answer. See below for the pool of specific topics and sample problems. The sample problems should give a good idea of what to expect. The proof problem will be one of the four listed here.

As a general rule, you can use facts that appear earlier in the book freely as needed. While any correct proof is of course acceptable, certainly the proofs as presented in the book will be adequate. The theorems listed here may be slightly different than what is in the book. Of course, you only need to prove what is asked by the problem.

You will each be allowed a 5×8 index card on which you can write whatever you think will be helpful.

Theorems:

1. Let $\mathcal{A}$ be a collection of open sets that cover $\mathbb{R}^n$. There exists a sequence $\phi_1, \phi_2, \ldots$ of smooth functions $\phi_i\colon \mathbb{R}^n \to \mathbb{R}$ (of class C^∞) such that the following are satisfied (let S_i denote the support of ϕ_i):
   (1) $\phi_i(x) \geq 0$ for all $x \in \mathbb{R}^n$ for each $i \in \mathbb{N}$.
   (2) Each set S_i is compact and contained in an element of $\mathcal{A}$.
   (3) Each point of A has a neighborhood that intersects finitely many of the sets S_i.
   (4) $\sum_{i=1}^{\infty} \phi_i(x) = 1$ for all $x \in \mathbb{R}^n$.
   Notes:
   - You may assume Lemma 16.1: for any rectangle $Q \subset \mathbb{R}^n$, there is a function $\phi\colon \mathbb{R}^n \to \mathbb{R}$ of class C^∞ such that $\phi(x) > 0$ for all $x \in \operatorname{Int} Q$ and $\phi(x) = 0$ otherwise.
   - Do not assume Lemma 16.2.
2. Let $A, B \subset \mathbb{R}^n$ be open sets and $g\colon A \to B$ be a diffeomorphism, where $n \geq 2$. For all $a \in A$, there is a neighborhood U_0 of a contained in A and a sequence of diffeomorphisms
   $$U_0 \xrightarrow{h_1} U_1 \xrightarrow{h_2} U_2 \longrightarrow \cdots \xrightarrow{h_k} U_k,$$
   where each $U_j \subset \mathbb{R}^n$ is open, such that the composite $h_k \circ \cdots \circ h_2 \circ h_1$ equals $g|U_0$ and such that each h_i is a primitive diffeomorphism.
3. **(CoV)** Let $g\colon A \to B$ be a diffeomorphism, where $A, B \subset \mathbb{R}^n$ are open sets. Then for every continuous function $f\colon B \to \mathbb{R}$ that is integrable over B, the function $(f \circ g)|\det Dg|$ is integrable over A and $\int_B f = \int_A (f \circ g)|\det Dg|$.
   Prove the inductive step of (CoV). Namely, assume that (CoV) holds in dimension $n-1$. Prove that (CoV) holds for each primitive diffeomorphism $f\colon U \to V$, where U, V are open sets in $\mathbb{R}^n$.
4. There is a unique function V that assigns to each k-tuple $(x_1, \ldots, x_k) \in (\mathbb{R}^n)^k$ a non-negative number such that
   - If $h\colon \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation, then $V(h(x_1), \ldots, h(x_k)) = V(x_1, \ldots, x_k)$.
   - If each $y_i \in \mathbb{R}^k \times \mathbf{0} \subset \mathbb{R}^n$, so that $y_i = (z_i, \mathbf{0})$ for some $z_i \in \mathbb{R}^k$, then $V(y_1, \ldots, y_k) = |\det[z_1 \cdots z_k]|$.

Short answer topic pool:

- smooth bump functions (Lemma 16.1), partition of unity
- change of variables, diffeomorphism, change of variables theorem
- polar coordinates, other computational examples
- diffeomorphisms preserve sets of measure zero
- primitive diffeomorphism
- parallelopiped, geometric meaning of determinant
- orientation, orientation-preserving maps

- orthogonal matrix, orthogonal group, classification of isometries of Euclidean space
- volume of a parallelopiped
- parametrized manifold, integral over a parametrized manifold (with respect to volume), volume of a parametrized manifold
- k-manifold, k-manifold without boundary, coordinate chart/patch
- examples and non-examples of manifolds
- interior and boundary of a manifold
- level sets/superlevel sets of functions of class C^r are manifolds (Theorem 24.4)
- integrating a scalar function over a manifold
- multilinear function, k-tensor
- elementary k-tensor, representation of tensors as linear combinations of elementary k-tensors, manipulation of tensors
- tensor product $\otimes$, its algebraic properties
- dual transformation T^*
- permutation, symmetric group, elementary permutation
- inversion, sign of a permutation
- alternating tensor, elementary alternating tensor
- wedge product $\wedge$, its algebraic properties, the map A
- tangent vector, velocity vector, tangent space, tangent bundle
- tensor field, differential form

Sample short answer problems:

1. Let $a = (1,0,3,1) \in \mathbb{R}^4$ and $b = (0,1,-1,2) \in \mathbb{R}^4$. What is the area of the parallelogram with vertices 0, a, b, and $a+b$?

2. Complete the following matrix A so that the linear tranformation $x \mapsto Ax$ is an isometry of $\mathbb{R}^3$.

$$A = \begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} & ? \\ 1/\sqrt{3} & 1/\sqrt{6} & ? \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$$

3. Define $\alpha\colon (-\pi,\pi) \to \mathbb{R}^2$ by $\alpha(t) = (\sin(t), \sin(2t))$. Let Y be the image of α. Is Y a 1-manifold having α as a coordinate chart? Justify your answer.

4. The standard torus T can be parametrized, up to a set of measure zero, by the map $\alpha\colon (0,2\pi)^2 \to \mathbb{R}^3$ defined by

$$\alpha(\theta,\varphi) = ((2+\cos\theta)\cos\varphi, (2+\cos\theta)\sin\varphi, \sin\theta).$$

Compute the area of T using this parametrization.

Note: In the follow problems, $\phi_1, \phi_2, \phi_3, \phi_4$ denote the elementary 1-tensors on $\mathbb{R}^4$, and $x = (x_1, x_2, x_3, x_4)$, $y = (y_1, y_2, y_3, y_4)$, $z = (z_1, z_2, z_3, z_4)$ denote points in $\mathbb{R}^4$.

5. Let f be the 3-tensor on $\mathbb{R}^4$ defined by $f(x,y,z) = 3x_1y_2z_3 - x_3y_3z_4$. Write f in terms of elementary 1-forms $\phi_1, \ldots, \phi_4$ and the tensor product $\otimes$.

6. Let $\psi_1 = \phi_1 \otimes \phi_4 \otimes \phi_3$ and $\psi_2 = \phi_1 \wedge \phi_4 \wedge \phi_3$. Evaluate $\psi_1(x,y,z)$ and $\psi_2(x,y,z)$, where $x = (1,0,1,0)$, $y = (0,1,0,1)$ and $z = (1,1,0,0)$.

7. Which of the following tensors on $\mathbb{R}^4$ are alternating?

(a) $f = \phi_1 + \phi_2$
(b) $g(x, y) = x_1 y_1$
(c) $h(x, y) = x_1 y_2 + x_1 y_3 - x_2 y_1 - x_3 y_1$

8. Let $n, k \in \mathbb{N}$. Give a formula for the dimension of $\mathcal{A}^k(\mathbb{R}^n)$, the space of alternating k-tensors on $\mathbb{R}^n$, with a short justification.