# MAT 322 MIDTERM 1 GUIDE

The test will consist of one proof from textbook/lecture (10 points), one unseen proof-based problem (10 points), and eight short answer problems worth a total of 20 points. The short answer problems will focus on definitions and statements of theorems, computations, relevant examples, homework problems, and the relationship between the various concepts. See below for the pool of specific topics and questions from last year's midterm. There are six theorems from the textbook that may appear on the test. I will pick two of these to put on the test, and you must prove one of them. On a separate file, you'll find sample unseen proof problems.

As a general rule for writing proofs, you can use facts that appear earlier in the book freely as needed. While any correct proof is of course acceptable, certainly the proofs as presented in the book will be adequate. In some cases, the theorems listed here are somewhat simpler than what is in the book. Of course, you only need to prove what is asked by the problem.

Each student will be a allowed a single $5 \times 8$ index card on which you can write whatever you believe will be helpful.

**Theorems:**

1. Let $A \subset \mathbb{R}^n$ be open. Let $f\colon A \to \mathbb{R}^m$ be a function such that the partial derivatives $D_j f_i$ of the component functions of $f$ exist at each point in $A$ and are continuous in $A$. Then $f$ is differentiable at each point of $A$.
2. Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$. Let $f\colon A \to \mathbb{R}^n$ and $g\colon B \to \mathbb{R}^p$ be functions, where $f$ satisfies $f(A) \subset B$. Let $a \in A$ and $b = f(a)$. If $f$ is differentiable at $a$, and if $g$ is differentiable at $b$, then the composite function $g \circ f$ is differentiable at $a$ with derivative
$$D(g \circ f)(a) = Dg(b) \cdot Df(a).$$
3. Let $A \subset \mathbb{R}^n$ be open and let $f\colon A \to \mathbb{R}^n$ be of class $C^1$. Let $B = f(A)$. Suppose that $Df(a)$ is non-singular for some $a \in A$. We assume the following two facts:
   - There is a neighborhood $U \subset A$ of $a$ and $\alpha > 0$ such that $|f(x_0) - f(x_1)| \geq \alpha |x_0 - x_1|$ for all $x_0, x_1 \in U$. In particular, $Df(x)$ is non-singular for all $x \in U$.
   - $f$ maps open sets in $U$ to open sets.

   Let $V = f(U)$ and let $g\colon V \to U$ be the inverse of $f|U$. Then $g$ is of class $C^1$ with derivative
$$Dg(y) = Df(g(y))^{-1}.$$
4. Let $A \subset \mathbb{R}^{k+n}$ be open and let $f\colon A \to \mathbb{R}^n$ be of class $C^r$. Write $f$ in the form $f(x, y)$ for $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$. Suppose that $(a, b)$ is a point of $A$ such that $f(a, b) = 0$ and
$$\det \frac{\partial f}{\partial y}(a, b) \neq 0.$$
   Then there is a neighborhood $B$ of $a$ in $\mathbb{R}^k$ and a unique continuous function $g\colon B \to \mathbb{R}^n$ such that $g(a) = b$ and $f(x, g(x)) = 0$ for all $x \in B$. The function $g$ is of class $C^r$.
5. Let $Q$ be a rectangle in $\mathbb{R}^n$ and let $f\colon Q \to \mathbb{R}$ be a bounded function. Let $D$ be the set of points of $Q$ at which $f$ fails to be continuous. Then $\int_Q f$ exists if and only if $D$ has measure zero in $\mathbb{R}^n$.
   - You may use the fact that $f$ is continuous at $a$ if and only if the oscillation $v(f; a) = 0$.
6. Let $Q = A \times B$, where $A$ is a rectangle in $\mathbb{R}^k$ and $B$ is a rectangle in $\mathbb{R}^n$. Let $f\colon Q \to \mathbb{R}$ be a bounded function; write $f$ in the form $f(x, y)$ for $x \in A$ and $y \in B$. If $f$ is integrable over $Q$, then $\underline{\int_{y \in B}} f(x, y)$ is integrable over $A$ as a function of $x$ and
$$\int_Q f = \int_{x \in A} \underline{\int_{y \in B}} f(x, y).$$

**Short answer topic pool:**

- Euclidean inner product
- rank of a matrix, invariance of rank under row operations
- Cauchy–Schwarz inequality
- axiomatic definition of determinant

- a matrix is invertible if and only if it has full rank and if and only if its determinant is nonzero
- Euclidean vs. sup metric
- interior/exterior/boundary points
- open and closed sets
- the closure of a set
- definitions of limits and continuity; basic properties
- the $\varepsilon$-neighborhood lemma
- compact sets; a set in $\mathbb{R}^n$ is compact if and only if it is closed and bounded
- uniform continuity
- differentiability vs. directional derivatives, partial derivatives
- computing derivatives
- determining differentiability/directional derivatives/continuity for individual functions
- functions of class $C^r$; differentiability assumptions needed for the various theorems
- computing derivatives with the chain rule
- computing derivatives by implicit differentiation
- formalism related to the Riemann integral
- recognizing Riemann integrable functions
- properties of measure zero sets
- evaluating an integral with Fubini's theorem
- definition of the integral on bounded regions; standard properties

**Sample short answer problems**

**1.** Which of the following is preserved by row operations applied to an $n \times m$ matrix $A$? (Choose all answers that apply.)

(a) The row rank of $A$ (the dimension of the subspace of $\mathbb{R}^m$ generated by the rows of $A$)
(b) The column rank of $A$ (the dimension of the subspace of $\mathbb{R}^n$ generated by the columns of $A$)

**2.** Let $A \subset \mathbb{R}^n$ be open and $f\colon A \to \mathbb{R}^n$ be a function. Mark **all** of the statements that are equivalent to "$f$ is continuous on $A$". (Choose all answers that apply.)

(a) For every open set $V \subset \mathbb{R}^n$, $f^{-1}(V)$ is open.
(b) For every open set $U \subset \mathbb{R}^n$, $f(U)$ is open.
(c) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x - y\| < \delta$ implies $\|f(x) - f(y)\| < \varepsilon$ for all $x, y \in A$.
(d) For every sequence $(x_n) \subset A$ converging to a point $x \in A$, $f(x_n)$ converges to $f(x)$.

**3.** Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^2$ and let $|\cdot|$ denote the sup norm. Sketch the sets $S = \{x \in \mathbb{R}^2 : \|x\| = 1\}$ and $T = \{x \in \mathbb{R}^2 : |x| = 1\}$ (the **unit sphere** in, respectively, the Euclidean metric and the sup metric). (Both $S$ and $T$ should be drawn on the same graph.)

**4.** Mark **all** of the statements that are true for **any** metric space $X$. (Choose all answers that apply.)

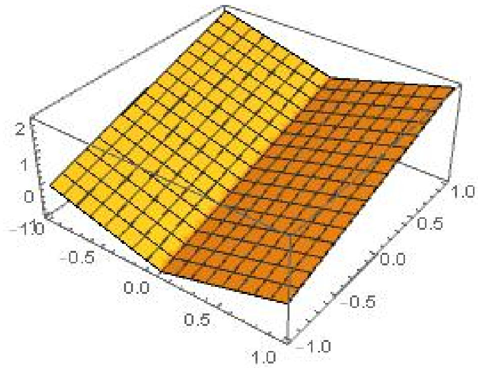
(a) If a set $A \subset X$ is compact, then it is closed and bounded.
(b) If a set $A \subset X$ is closed and bounded, then it is compact.
(c) If a set $A \subset X$ is compact and $f\colon A \to \mathbb{R}$ is continuous, then $f$ has a maximum value.
(d) If a set $A \subset X$ is compact and $f\colon A \to \mathbb{R}$ is continuous, then $f$ is uniformly continuous.

**5.** Define the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x) = \begin{cases} \frac{x^2y^2}{x^2y^2+(x-y)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

For which nonzero vectors $u \in \mathbb{R}^2$ does the directional derivative $g'(0; u)$ exist?

**6.** Let $f\colon [0,1]^2 \to \mathbb{R}$ be the function plotted here. Which correctly describes $f$? (Choose one.)




(a) There are some points at which $f$ is not differentiable.
(b) $f$ is differentiable at every point but not of class $C^1$.
(c) $f$ is of class $C^1$.

**7.** Let $f, g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by the formulas

$$f(x_1, x_2) = (e^{2x_1+x_2}, 3x_2 - \cos(x_1))$$
$$g(y_1, y_2) = (3y_1 + 2y_2, y_1^2 + 1)$$

Let $F = g \circ f$. Compute $DF(0)$.

**8.** Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a function of class $C^1$ such that $f(3, -1, 2) = 0$ and

$$Df(3, -1, 2) = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Let $g\colon (3-\varepsilon, 3+\varepsilon) \to \mathbb{R}^2$ be such that $g(3) = (-1, 2)$ and $f \circ g((3-\varepsilon, 3+\varepsilon)) = 0$ as given by the implicit function theorem, for some small $\varepsilon > 0$. Find $Dg(3)$.

**9.** Let $Q = [0, 1]$. Define the function $f\colon Q \to \mathbb{R}$ by $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$. Determine $\underline{\int_Q} f$ and $\overline{\int_Q} f$.

**10.** Write out the definition of a **set of measure zero in $\mathbb{R}^n$**.