MIDTERM 1 PRACTICE KEY

1. Let i_U and i_V denote the identity map on U and V, respectively. According to the Chain Rule, $I_n = Di_U = D(F^{-1}) \cdot DF$ and $I_m = Di_V = DF \cdot D(F^{-1})$. So DF has both a left and right inverse, which implies that DF is a square matrix. Thus $m = n$.

[Note: Theorem 8.2 in the book includes a similar claim, but it is only assumed that DF is non-singular rather than the assumption here that F^{-1} exists and is C^1. See Theorem 2.2 for the claim regarding DF being a square matrix.]

2. For the first part, we want to show that $f(x, y_1) = f(x, y_2)$ for all $x, y_1, y_2 \in \mathbb{R}$, given that $\partial y f = 0$ everywhere. We may assume that $y_1 < y_2$. Consider the function $\phi\colon [y_1, y_2] \to \mathbb{R}$ defined by $\phi(t) = f(x, t)$. Then $\phi'(t) = \partial_y(x, t)$. According to the mean value theorem, there exists $c \in (y_1, y_2)$ such that $\phi'(c) = (\phi(y_2) - \phi_(y_1))/(y_2 - y_1)$. By assumption $\phi'(c) = 0$, and therefore $\phi(y_1) = \phi(y_2)$. This establishes the first claim.

We now assume that in addition $\partial_x f = 0$. The same argument with the roles of x and y reversed shows that $f(x_1, y) = f(x_2, y)$ for all $x_1, x_2, y \in \mathbb{R}$. Then we have for all $x_1, y_1, x_2, y_2 \in \mathbb{R}$ $f(x_1, y_1) = f(x_2, y_1) = f(x_2, y_2)$. It follows that f is constant.

For the final part, we may take the function f defined by

$$f(x, y) = \begin{cases} x^2 & \text{if } x > 0 \text{ or } y > 0 \\ -x^2 & \text{if } x < 0 \text{ and } y < 0 \end{cases}.$$

Then $\partial_y f = 0$. Moreover, $\partial_x f = 2x$ if $x > 0$ or $y > 0$ and $\partial_x f = -2x$ if $x < 0$ and $y < 0$. This is continuous on the domain A. Finally, $f(-1, 1) = 1$ but $f(-1, -1) = -1$, so f is not independent of x.

3. We identify $\mathbb{R}^{n(n+1)/2}$ with a symmetric matrix in the natural way. For each $i, j \in \{1, \ldots, n\}$ with $i \le j$, we have corresponding partial derivative $D_{ij} \det(I)$. Together these partial derivatives comprise the derivative $D \det(I)$. We claim that $D_{ij} \det(I) = 1$ if $i = j$ and $D_{ij} \det(I) = 0$ otherwise.

Let S_{ij} denote the symmetric matrix that has (i, j)-th and (j, i)-th entry equal to 1 and is zero otherwise. Then

$$D_{ij} \det(I) = \lim_{t \to 0} \frac{\det(I + tS_{ij}) - \det(I)}{t}.$$

Note that $\det(I) = 1$, while $\det(I + tS_{ij}) = 1 + t$ if $i = j$ and $\det(I + tM_{ij}) = 1 - t^2$ if $i \neq j$. The claim follows.

Bonus: The same conclusion holds, though the argument is slightly different. Here we take M_{ij} to denote the matrix that has (i, j)-th entry equal to 1 and is zero otherwise. Note that $\det(I + tM_{ij}) = 1$ if $i \neq j$, so the difference quotient vanishes.

[Note: I didn't see any way to use the hint. In principle, it could be helpful since a diagonalizable matrix A can be written as $A = P^{-1}DP$, where D is diagonal, so that $\det(A) = \det(D)$. But the determinant above already seems straightforward enough.]

4. (a) f is integrable if and only if the set D of points in Q on which f is not continuous has measure zero. This means for all $\varepsilon > 0$ there exists a countable collection $\{Q_1, Q_2, \ldots\}$ of rectangles covering D such that $\sum_{i=1}^{\infty} v(Q_i) < \varepsilon$, where $v(Q_i)$ is the n-dimensional volume of Q_i.

(b) Let $x \in Q \setminus Z$. Since $Q \setminus Z$ is open, there is a neighborhood of x on which f is identically zero. Thus f is continuous at x. It follows that the set D of points of discontinuity of f in Q is contained in Z and thus has measure zero. By (a), we see that f is integrable on Q.

(c) Define f by $f(x_1, \ldots, x_n) = 1$ if all coordinates x_i are rational and $f(x_1, \ldots, x_n) = 0$ otherwise. Then $f = 0$ except on a countable set, which necessarily has measure zero (since a single point has measure zero, and the countable union of sets of measure zero has measure zero). However, f is not integrable, since $f = 1$ on a dense set.

5. We claim that

$$\partial_i f(x) = \int_{\widehat{Q}_i} f,$$

where $\widehat{Q}_i = [a_1, x_1] \times \cdots \times [a_{i-1}, x_{i-1}] \times \cdots \times [a_{i+1}, x_{i+1}] \times \cdots \times [a_n, x_n]$. For notational simplicity, we assume that $i = n$ and write $\widehat{Q} = \widehat{Q}_n$. It is enough to show that the right-hand side equals

$$\lim_{t \to 0} \frac{1}{t} \int_{\widehat{Q} \times [x_n, x_n + t]} f.$$

Apply Fubini's theorem to write this as

$$\lim_{t \to 0} \frac{1}{t} \int_{\widehat{Q}} \int_{x_n}^{x_n + t} f$$

Denote the value on the previous line by L. Let $\varepsilon > 0$. Note that f is uniformly continuous on Q; therefore there exists $\delta > 0$ such that $|f(y, x_n + t) - f(y, x_n)| < \varepsilon$ for all $y \in \widehat{Q}$, $t \in (-\delta, \delta)$. For fixed $y \in \widehat{Q}_i$ we thus have

$$t(f(y, x_n) - \varepsilon) \leq \int_{x_i}^{x_i + t} f \leq t(f(y, x_n) + \varepsilon)$$

for all $t \in (-\delta, \delta)$. This implies

$$\int_{y \in \widehat{Q}} (f(y, x_n) - \varepsilon) \leq L \leq \int_{y \in \widehat{Q}} (f(y, x_n) + \varepsilon)$$

and thus

$$\int_{y \in \widehat{Q}} f(y, x_n) - \varepsilon v(\widehat{Q}) \leq L \leq \int_{y \in \widehat{Q}} f(y, x_n) + \varepsilon v(\widehat{Q})$$

Letting $\varepsilon \to 0$ gives $L = \int_{\widehat{Q}} f$ as claimed.

HW p. 79 4(c). [Note: I'm doing this from scratch without using the results of part (b). Take my solution with some caution; there's a chance of some arithmetic mistake, and my solution doesn't agree with any of the answers submitted by the class. None of the solutions from the class agreed with each other either, with the exception of two students that obtained $-743/72$. My work also gives the solution $-743/72$ if the "56" below is replaced by "67", which these students had at the same step. I haven't been able to reconcile the difference yet.]

Let $H \colon \mathbb{R}^3 \to \mathbb{R}^2$ denote the function precomposed with F in the definition of G. We also write $w = F(u, v) = G(x, y, z)$ and $(u, v) = H(x, y, z)$. For conciseness and simplicity, we follow "variable notation" even though there is some ambiguity present. According to the implicit function theorem, we have

$$Dg = [-(\partial_z w)^{-1}\partial_x w \quad -(\partial_z w)^{-1}\partial_y w].$$

From here, it's a tedious repeated application of the product rule and chain rule until we can isolate D_2D_1g. The product rule gives

$$D_2D_1g = \partial_y\left(-(\partial_z w)^{-1}\partial_x w\right) = -(\partial_z w)^{-1}\partial_y\partial_x w + (\partial_z w)^{-2}\partial_y\partial_z w \cdot \partial_x w.$$

Next, we have

$$\begin{aligned}\partial_x w =& \partial_u w \cdot \partial_x u + \partial_v w \cdot \partial_x v\\ \partial_y w =& \partial_u w \cdot \partial_y u + \partial_v w \cdot \partial_y v\\ \partial_z w =& \partial_u w \cdot \partial_z u + \partial_v w \cdot \partial_z v\end{aligned}$$

and then

$$\begin{aligned}\partial_y\partial_x w =& \partial_y\partial_u w \cdot \partial_x u + \partial_u w \cdot \partial_y\partial_x u + \partial_y\partial_v w \cdot \partial_x v + \partial_v w \cdot \partial_y\partial_x v\\ =& (\partial_u\partial_u w \cdot \partial_y u + \partial_v\partial_u w \cdot \partial_y v) \cdot \partial_x u + \partial_u w \cdot \partial_y\partial_x u\\ &+ (\partial_u\partial_v w \cdot \partial_y u + \partial_v\partial_v w \cdot \partial_y v) \cdot \partial_x v + \partial_v w \cdot \partial_y\partial_x v\\ \partial_y\partial_z w =& \partial_y\partial_u w \cdot \partial_z u + \partial_u w \cdot \partial_y\partial_z u + \partial_y\partial_v w \cdot \partial_z v + \partial_v w \cdot \partial_y\partial_z v\\ =& (\partial_u\partial_u w \cdot \partial_y u + \partial_v\partial_u w \cdot \partial_y v) \cdot \partial_z u + \partial_u w \cdot \partial_y\partial_z u\\ &+ (\partial_u\partial_v w \cdot \partial_y u + \partial_v\partial_v w \cdot \partial_y v) \cdot \partial_z v + \partial_v w \cdot \partial_y\partial_z v\end{aligned}$$

Finally, note that

$$DH = \begin{bmatrix} 1 & 2 & 3\\ 3x^2 & 2y & -2z\end{bmatrix}.$$

At this point, we have the formulas we need on hand. We just need to evaluate them at the point $(x, y, z) = (-2, 3-1)$, corresponding to the values $(u, v) = (0, 0)$ and $w = 0$.

We have: $\partial_x u = 1$, $\partial_y u = 2$, $\partial_z u = 3$, $\partial_x v = 12$, $\partial_y v = 6$, $\partial_z v = 2$. From the information in the problem, we have $\partial_u w = 2$, $\partial_v w = 3$, $\partial_u\partial_u w = 3$, $\partial_u\partial_v w = -1$, $\partial_v\partial_v w = 5$. The second partials for u, v are zero except for $\partial_x\partial_x v = 6$, $\partial_y\partial_y v = 2$, $\partial_z\partial_z v = -2$. Thus we have

$$\begin{aligned}\partial_y\partial_x w =& (3\cdot 2 + (-1)\cdot 6)\cdot 1 + 2\cdot 0\\ &+ (-1\cdot 2 + 5\cdot 6)\cdot 12 + 3\cdot 0\\ =& 336\\ \partial_y\partial_z w =& (3\cdot 2 + (-1)\cdot 6)\cdot 3 + 2\cdot 0\\ &+ (-1\cdot 2 + 5\cdot 6)\cdot 2 + 3\cdot 0\\ =& 56\end{aligned}$$

and

$$\begin{aligned}\partial_x w =& 2\cdot 1 + 3\cdot 12 = 38\\ \partial_z w =& 2\cdot 3 + 3\cdot 2 = 12.\end{aligned}$$

Finally, this gives

$$D_2D_1g = -12^{-1}\cdot 336 + 12^{-2}\cdot 56\cdot 38 = -119/9$$