MAT 203 LECTURE OUTLINE 10/13

- We are starting Chapter 14 of the textbook on integrating multivariable functions. You'll likely find the material of the last two chapters to be more conceptually difficult than the first half of the course. On the other hand, the course calendar is arranged to go at a slower pace.
- Today we will look at two flavors of integrals: *iterated integrals* and *double integrals*. The relation between these is given by *Fubini's theorem*.
- Let's recall a bit of Calculus 1. In that class, the definite integral of a function is defined as the limit of Riemann sums of that function. Recall that a *Riemann sum* of a function $f\colon [a,b] \to \mathbb{R}$ is an expression of the form
$$\sum_{j=1}^{n} f(a + j\Delta x)\Delta x,$$
where $\Delta x = (b-a)/n$ and n is some positive integer. The Riemann sum is an approximation to the signed area of the region bounded by $x = a$, $x = b$, the graph of f and the x-axis. ("signed" means that any part below the x-axis is counted as negative area.) As n is chosen to be large, then the Riemann sum becomes an arbitrarily good approximation to the signed area of this region (if f is piecewise continuous). Then we define the *definite integral* of f to be
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{j=1}^{n} f(a + j\Delta x)\Delta x.$$
As we just discussed, the intuitive interpretation of the definite integral is "area under the curve", or more accurately "signed area".
- The definite integral can be evaluated directly as a limit. However, it is usually much more convenient to use the **Fundamental Theorem of Calculus**, which states that a definite integral can be evaluated by finding an antiderivative of the function. Precisely, if F is an antiderivative of f (that is, if $F'(x) = f(x)$), then
$$\int_a^b f(x)\,dx = F(b) - F(a).$$
Recall that much of Calculus 2 is dedicated to techniques of integration, or, perhaps better said, techniques for finding antiderivatives.
- We want to develop the same idea for multivariable functions. We can think of the definite integral as now representing "volume under a surface". This is called a *double integral*. Instead of dividing an interval into smaller intervals as we did for the Riemann sum in the definite integral, now we divide a rectangle into smaller rectangles. Consider a function $f(x,y)\colon R \to \mathbb{R}$ defined on a planar rectangle $R = [a,b] \times [c,d]$. The *double integral* of f is
$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x + i\Delta x, y + i\Delta y)\Delta x\,\Delta y,$$
where $\Delta x = (b-a)/m$ and $\Delta y = (d-c)/n$. See Figures 14.8-14.11 in the textbook for visuals. In this formula, $\iint$ is the symbol for double integral, which suggests "integrating over two variables". The A in "dA" stands for *area*. We say the function $f(x,y)$ is *integrable* if the above limit exists.
- We can also define the double integral for functions $f(x,y)$ defined on bounded regions R other than rectangles. This can be done by using a Riemann sum comprising those rectangles contained within R. It's the latter approach that's described in the textbook.
- Now we get to the question of evaluating a double integral. Fubini's theorem states that this can be done using an iterated integral. Iterated integrals are loosely analogous to taking partial derivative: you treat one variable as a constant and integrate with respect to the other one. In our situation, you

integrate twice: once with respect to x and once with respect to y. It's easiest to get a feel for this by doing examples.

- Example. Compute $\int_1^2 \int_1^x (2xy + 3y^2) dy\, dx$. (The answer is 5.)
- Notice how the limits of integration represent a planar domain R, in this case the triangle with vertices $(1,1)$, $(2,1)$, $(2,2)$. The same integral can be represented two ways: the inner integral with respect to y and the outer integral with respect to x, or the inner integral with respect to x and the outer integral with respect to y. Depending on the problem, one way may be easier to evaluate than another. If you feel so inclined, try rewriting the previous integral with the order of integration reversed and check that you get the same answer.
- Typically, we consider regions R that are one of two types: *vertically simple* and *horizontally simple*. The first means that R is represented by an integral of the form $\int_a^b \int_{g_1(x)}^{g_2(x)} \ldots\, dy\, dx$. The second means that R is represented by an integral of the form $\int_c^d \int_{h_1(y)}^{h_2(y)} \ldots\, dx\, dy$. If we integrate the function 1 over R, we get the area of R. This can also be thought of as the volume of a cylinder of height one over R. To handle more complicated regions, we can divide them into multiple regions, each of which is vertically or horizontally simple.
- We can now give a statement of Fubini's theorem. Suppose first that R is vertically simple: that is, $R = \{(x,y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}$ for some $a < b$ and continuous functions $g_1, g_2 \colon [a,b] \to \mathbb{R}$. Then
$$\iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) dy\, dx.$$
Suppose next that R is horizontally simple: that is, $R = \{(x,y) : c \le y \le d, h_1(y) \le x \le h_2(x)\}$ for some $a < b$ and continuous functions $g_1, g_2 \colon [a,b] \to \mathbb{R}$. Then
$$\iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) dy\, dx.$$
- Example. $\int_0^2 \int_{y^2}^4 dx\, dy$. Compute this integral directly, and by switching the order of integration. (The answer is 16/3.)
- Example. Find the volume of the region in $\mathbb{R}^3$ bounded by $z = 0$, $y = 0$, $y = x$, $x = 1$, and the $z = f(x,y) = e^{-x^2}$. (The answer is $(e-1)/(2e)$.)

  This problem does not explicitly tell you to integrate, but recall that the interpretation of double integrals is "volume under a surface", in this case the surface $f(x,y) = e^{-x^2}$ over the planar region bounded by $y = 0$, $y = x$, $x = 1$.

  There are two ways to set up this integral, depending on whether we treat the domain as a vertically simple or horizontally simple region. You'll find that the problem is quite straightforward with one way and basically impossible to do the other way. Try both and see which one works.