MAT 203 LECTURE OUTLINE 10/18

- We have two topics today: integration in polar coordinates (Section 14.3) and applications of integration to finding centers of mass and moments of inertia (Section 14.4).
- Let's start with polar coordinates. This is our first example of a "change of variables", an idea we'll look at more systematically in section 14.8. Recall that polar coordinates are defined by the relations $x = r\cos(\theta)$, $y = r\sin(\theta)$.
- To get to the punchline, the conversion between a standard integral (i.e., in rectangular coordinates) of a function $f(x, y)$ over a region R and an integral in polar coordinates is
$$\iint_R f(x,y)\,dA = \iint_{R'} f(r\cos(\theta), r\sin(\theta))\,r\,dA,$$
where R' is the region of the (r, θ)-plane corresponding to R. To understand the derivation of this formula, use Figures 14.24-14.27 in the textbook as a guide. To briefly summarize here: the idea is to form a Riemann sum in polar coordinates, with the area of each polar rectangle given by $\Delta A_i = r_i \Delta r_i \Delta\theta_i$. This can be derived by computing the area of a sector, which we did in class.
- One step in doing an integral in polar coordinates is to describe the region of integration in terms of polar coordinates. This takes a bit of practice. See Figure 14.23 in the textbook for typical regions. These regions are usually well-behaved with respect to polar coordinates, such as disks, sectors, and annuli (i.e., ring-shaped regions).
- Similarly to horizontally and vertically simple regions, we can talk about r-*simple* and θ-*simple* regions of integration. An r-simple region has the form $\{(r, \theta) : \alpha \leq \theta \leq \beta, 0 \leq g_1(\theta) \leq r \leq g_2(\theta)\}$ for some constants α, β and functions $g_1(\theta), g_2(\theta)$. In this case the formula for integration in polar coordinates becomes
$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta.$$
For a θ-simple region, we have the formula
$$\iint_R f(x,y)\,dA = \int_{r_1}^{r_2} \int_{h_1(r)}^{h_2(r)} f(r\cos(\theta), r\sin(\theta))\,r\,d\theta\,dr.$$
- Example. $\displaystyle\iint_R (x^2 + y)\,dA$, where R is the upper half-disk of radius 2 in the plane.

  The setup is
$$\iint_R (x^2+y)\,dA = \int_0^\pi \int_0^2 (r^2\cos^2(\theta) + r\sin(\theta))r\,dr\,d\theta,$$
which integrates to $2\pi + 16/3$.
- Example. $\displaystyle\iint_R x\,dA$, where R is the disk of radius 1 centered at $(1, 0)$.

  The setup for this problem is more difficult, since it requires finding a formula for the boundary circle $(x-1)^2 + y^2 = 1$ in polar coordinates. This formula is $r = 2\cos(\theta)$, where $0 \leq r \leq \pi$. (not $0 \leq r \leq 2\pi$). The integral in polar coordinates is then
$$\iint_R x\,dA = \int_0^\pi \int_0^{2\cos(\theta)} r\cos(\theta) r\,dr\,d\theta.$$
To evaluate this integral, you need to use the identity $\cos^2(\alpha) = (1 + \cos(2\alpha))/2$ twice. The answer is π.
- Now, let's look at the applications. We've already discussed the basic physical interpretation of double integrals as "volume under a surface". These applications will build on this idea.

- We consider a thin object with variable material or density (the book calls such an object a *lamina*). We represent such a lamina as a region R in the planar with with planar coordinates (x, y) and a function $\rho(x, y)$ representing the mass density. The first problem is to find the total mass of a lamina. This is done simply by integrating the density ρ over R.
- The next problem, and a somewhat more difficult one, is finding the center of mass of a lamina. Physically, if you were to hold up the lamina with one finger in such a way that the lamina doesn't fall over, you would place your finger at the center of mass.
- In simple cases such as objects of constant density with nice symmetry, the solution to finding the center of mass is obvious. For example, the center of mass of a disk or square of constant density is the center point of the disk or square.
- In general, the problem can be solved by computing an integral. The idea is as follows. Denote the center of mass by $(\bar{x}, \bar{y})$. To find $\bar{x}$, we integrate the function x weighted by the density ρ, then divide by the total mass m: $\bar{x} = \frac{1}{m}\iint_R x\,\rho(x, y)\,dA$. The quantity $M_y = \iint_R x\,\rho(x, y)\,dA$ is called the *moment of mass* with respect to the y-axis and is denoted in the textbook by M_y. There is a general concept in math and physics called a *moment* meaning "product of a physical quantity and a distance", of which this is an example, though we won't worry much about this for the purpose of this class. We find $\bar{y}$ similarly: $\bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_R y\,\rho(x, y)\,dA$, where $M_x = \iint y\,\rho(x, y)\,dA$ is called the moment of mass with respect to the x-axis.
- Example. Find the center of mass of a lamina the shape of the region under the parabola $4 - x^2$ and above the x-axis. Assume a constant density 1.

  First, the mass is $\int_{-2}^{2}(4 - x^2)\,dx = 32/3$.

  Note by symmetry that $\bar{x} = 0$.

  Next, moment of mass with respect to the x-axis is $M_x = \int_{-2}^{2}\int_{0}^{4-x^2} y\,dy\,dx = 256/15$. So $\bar{y} = M_x/x = 8/5$.
- There are also *second moments* (or *moments of inertia*), which we will just touch on briefly. These are found by integrating x^2 or y^2 weighted by the density $\rho(x, y)$ (instead of integrating x or y). These are
  - $I_y = \iint x^2\rho(x, y)\,dA$ (second moment with respect to the y-axis)
  - $I_x = \iint y^2\rho(x, y)\,dA$ (second moment with respect to the x-axis)
  - The sum $I_x + I_y$ is called the *polar moment of inertia* and is denoted by I_0.