MAT 203 LECTURE OUTLINE 8/23

- This course is about **multivariable calculus**. For the first couple weeks, we will focus on the "multivariable" part of this name before getting to the "calculus" part.
- Notation:
  — $\mathbb{R} = \mathbb{R}^1$ is the set of real numbers, or *real line*
  — $\mathbb{R}^2$ is the *plane*, or *Euclidean plane* to emphasize that the distance between two points is given by the distance formula
  — $\mathbb{R}^3$ is *space*, or 3-*space*, or *Euclidean* 3-*space*, or 3-*dimensional Euclidean space*
  — $\mathbb{R}^n$ is n-*space*, or *Euclidean* n-*space*, or n-*dimensional Euclidean space*
  — elements of these sets are called "points". In the context of multivariable calculus, an element of $\mathbb{R}$ is often called a *scalar*.
  — there are various common notations for points, including:
    — (x, y), $x = (x_1, x_2)$, $P = P(p_1, p_2)$ for points in $\mathbb{R}^2$
    — (x, y, z), $x = (x_1, x_2, x_3)$, $P = P(p_1, p_2, p_3)$ for points in $\mathbb{R}^3$
    — $x = (x_1, x_2, \ldots, x_n)$ for points in $\mathbb{R}^n$
  — $\in$ is the symbol for set membership, e.g., "Let $x \in \mathbb{R}^3$."
- Most of multivariable calculus applies to n dimensions for any postive integer n. However, we will focus most on the case of dimension 2 and 3.
- Given two points $P = P(p_1, p_2, \ldots, p_n), Q = Q(q_1, q_2, \ldots, q_n$ in $\mathbb{R}^n$, we can consider the *directed line segment* from P to Q, denoted by $\overrightarrow{PQ}$. Here, P is called the *initial point* and Q is called the *terminal point.*
- The directed line segment $\overrightarrow{PQ}$ has a *magnitude*, or length, denoted by $\|\overrightarrow{PQ}\|$ and given by the distance formula

$$\|\overrightarrow{PQ}\| = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + \cdots + (p_n - q_n)^2}.$$

  It also has a *direction*, which is often described using angles in some coordinate system.
- A *vector* is a set of **equivalent** directed line segments, where "equivalent" means having the same magnitude and direction (but possibly different initial point). In practice, we usually don't distinguish between a vector and a directed line segment that represents it.
- A vector is in *standard position* if its initial point is the origin.
- Vectors are often denoted using boldface (in print), with arrows (when written by hand), with angled brackets (when written in component form): $\mathbf{u}$, $\vec{u}$, $\langle u_1, u_2\rangle$, $\langle u_1, u_2, u_3\rangle$, $\langle u_1, u_2, \ldots, u_n\rangle$.
- Example. Write out the vector from $P(1, 2)$ to $Q(4, 3)$ in component form. Draw the vector in standard position. What is $\|\overrightarrow{PQ}\|$?
- The *standard unit vectors* in $\mathbb{R}^2$ are $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$. The *standard unit vectors* in $\mathbb{R}^3$ are $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$ and $\mathbf{k} = \langle 0, 0, 1\rangle$. Though the notation is less compact, it is often convenient when doing computations to write out vectors in terms of the standard unit vectors (a *weighted sum* or *linear combination*). For example, $\langle 2, 3, -1\rangle = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.
- Vectors can also be described using angles. In two dimensions, this can be done by measuring the angle counterclockwise from the positive real axis (the preferred way for a mathematician). For example, talking about an angle of 60° (or $\pi/3$ [radians]) would likely by default be measured counterclockwise from the positive real axis. However, in a navigational setting, this same direction can be expressed using the cardinal directions as "bearing of 30 degrees" or "N 30° E" (30 degrees east of north).
- Two vector operations: addition and scalar multiplication. These are defined in the expected way. Let $\mathbf{u} = \langle u_1, u_2, \ldots, u_n\rangle$ and $\mathbf{v} = \langle v_1, v_2, \ldots, v_n\rangle$, and $c \in \mathbb{R}$. Then
  — $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n\rangle$
  — $c\mathbf{u} = \langle cu_1, cu_2, \ldots, cu_n\rangle$.
- Similarly, we define

$$-\mathbf{u} = (-1)\mathbf{u} = \langle -u_1, -u_2, \ldots, u_n\rangle$$

  and

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \langle u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n\rangle.$$

- Vectors often represent a force acting on an object. If multiple forces act on the same object, then these can be added together to give the net force or *resultant force*. In physics, there are various words (for example, *tension*) referring to different forces, but mathematically these behave identically.
- Example (Exercise 11.1.79 in the book, which has a diagram). Two ropes beginning at points A and B on the ceiling meet at a third point C. From C hangs a 3000 lb. weight. The rope from A makes an angle of $50°$ with the ceiling; the rope from B makes an angle of $30°$ with the ceiling. Find the tension in the two ropes.

  The point of the problem is that, since the configuration is not accelerating, all the forces must sum to zero. Break up all the forces into vertical and horizontal components, then add.
- Class question 1. Let $\mathbf{v}$ be a non-zero vector. What is $\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\|$?
- Class question 2. If $\|\mathbf{u}\| = 2$ and $\|\mathbf{v}\| = 3$, what is the largest possible value of $\|\mathbf{u}+\mathbf{v}\|$? [The fact applied here is the *triangle inequality*.]