MAT 203 LECTURE OUTLINE 8/25

- We look at two more vector operations today: the **dot product** and the **cross product**. Both take two vectors as input; the dot product gives a scalar as output, while the cross product gives a vector.
- Use the notation $\mathbf{u} = \langle u_1, u_2, \dots, u_n \rangle$, $\mathbf{v} = \langle v_1, v_2, \dots, v_n \rangle$, depending on the dimension we are working in.
- The *dot product* is defined as
$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$
It is also called the *inner product* or *scalar product*.
- The dot product has a geometric meaning seemingly unrelated to this definition. If we take $\mathbf{w}$ to be the *projection* of $\mathbf{u}$ onto $\mathbf{v}$ (we'll explain more about projections below), then
$$|\mathbf{u} \cdot \mathbf{v}| = \|\mathbf{w}\| \, \|\mathbf{v}\|.$$
The dot product is positive if $\mathbf{u}$ and $\mathbf{v}$ point in the same direction and negative if they point in opposite directions. We can think of the dot product as "multiplying the magnitudes of the vectors, weighted according to how much they point in the same direction".
- An equivalent formula, and one that you should make sure to remember, is
$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta),$$
where θ is the angle made by the two vectors when positioned with the same initial point. This is Theorem 11.5 in the book, where a proof is also given. This formula follows from the identity $\mathbf{w} \cdot \mathbf{w} = \|\mathbf{w}\|^2$ and is essentially a statement of the **law of cosines**.
- Another basic observation is that $\mathbf{u} \cdot \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (perpendicular)
- The vector $\mathbf{u}$ can be written uniquely as the sum of a vector parallel to $\mathbf{v}$ (called the *projection* of $\mathbf{u}$ onto $\mathbf{v}$, written as $\mathbf{proj}_{\mathbf{v}}\,\mathbf{u}$) and a vector orthogonal to $\mathbf{v}$. The projection is given by the formula
$$\mathbf{proj}_{\mathbf{v}}\,\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$$
and satisfies
$$\|\,\mathbf{proj}_{\mathbf{v}}\,\mathbf{u}\| = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|}.$$
- To motivate the cross product, consider this problem: given two non-zero vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^3$, not pointing in the same direction. Find a vector $\mathbf{w}$ in $\mathbb{R}^3$ that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. (Note that any two solutions are scalar multiples of each other.)
- If you've taken linear algebra, you've encountered problems of this type. The two constraints on $\mathbf{w}$ (that $\mathbf{w}$ is orthogonal to $\mathbf{u}$ and that $\mathbf{w}$ is orthogonal to $\mathbf{v}$) each give a linear equation satisfied by $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$. So we have a system of two linear equations in three variables, thus a one-dimensional space of solutions (since $1 = 3 - 2$). So the previous problem is something that can be solved by linear algebra.
- It turns out that the solution takes an elegant form, which is the cross product. It is defined by
$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - v_2u_3)\mathbf{i} - (u_1v_3 - v_1u_3)\mathbf{j} + (u_1v_2 - v_1u_2)\mathbf{k}.$$
This can be written in determinant notation as
$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}.$$
(This is really just a mnemonic. If you know linear algebra, determinant notation is familiar; otherwise, you can ignore this.) Note that the cross product is only defined in dimension 3.
- Check by taking $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u}$ and $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v}$ that the cross product $\mathbf{u} \times \mathbf{v}$ really is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
- The direction that $\mathbf{u} \times \mathbf{v}$ points is determined by the *right-hand rule*: point your hand in the direction of $\mathbf{u}$, then curl your fingers towards $\mathbf{v}$. The direction your thumb is pointing is the direction of $\mathbf{u} \times \mathbf{v}$.

- The magnitude of $\mathbf{u} \times \mathbf{v}$ satisfies
$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin(\theta),$$
where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. This is the same as the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$. This property accounts for much of the usefulness of the cross product in applications. See Theorem 11.8 in the textbook for the proof.
- Note that the cross product is antisymmetric: $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$. This makes sense when you think about the right-hand rule.
- Class question 1. What is $\mathbf{i} \times \mathbf{k}$?
- Class question 2. What is $\|(\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + \mathbf{k})\|$?
- These problems may be done directly from the definition of cross product or by using the relationships $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$, $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$, $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$. Notice that the cross product is positive whenever $\mathbf{i}, \mathbf{j}, \mathbf{k}$ appear in *increasing* cyclic order and negative whenever they appear in *decreasing* cyclic order.