MAT 203 LECTURE OUTLINE 9/13

- This is the first lecture which I expect to be genuinely new material for most of the class. We will develop a framework for understanding the motion of a particle in $\mathbb{R}^3$.
- Warmup: how do you take the derivative $\frac{d}{dt}\|\mathbf{r}(t)\|$?
- Let $\mathbf{r}$ be a vector-valued function tracing a smooth curve C. Recall that the derivative $\mathbf{r}'(t)$ is tangent to the curve C at the point $\mathbf{r}(t)$. A different parametrization of the curve C would produce tangent vectors of different length. To account for this, we define the *unit tangent vector*
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$
whenever $\mathbf{r}'(t) \neq \mathbf{0}$.
- Next, the *principal unit normal vector* at t is defined as
$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$
Take a moment to think about what this represents geometrically. Since this formula contains the derivative of the unit tangent vector, we expect $\mathbf{N}(t)$ to be related to the second derivative $\mathbf{r}''(t)$, that is, the acceleration of a moving particle with position $\mathbf{r}(t)$. We will soon make this relationship precise.
- We can check that $\mathbf{N}(t)$ is orthogonal to $\mathbf{T}(t)$, as the name "normal vector" suggests. Why does this make sense geometrically?
- It is a theorem that the acceleration vector $\mathbf{r}''(t)$ lies in the plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$. This essentially follows from the product rule. If $\|\mathbf{r}'(t)\|$ is constant, then in fact $\mathbf{r}''(t)$ is a scalar multiple of $\mathbf{N}(t)$. In general, we can define the *tangential* and *normal components* of acceleration as $\mathbf{r}''(t) \cdot \mathbf{T}(t)$ and $\mathbf{r}''(t) \cdot \mathbf{N}(t)$, respectively. We can derive the formulas
$$a_{\mathbf{T}} = \frac{d}{dt}\|\mathbf{r}(t)\| = \mathbf{r}''(t) \cdot \mathbf{T}(t) = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|}$$
and
$$a_{\mathbf{N}} = \|\mathbf{r}'(t)\|\|\mathbf{T}'(t)\| = \mathbf{r}''(t) \cdot \mathbf{N}(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}.$$
- For a curve C parametrized by a vector-valued function $\mathbf{r}(t)$, the *curvature* K at a point t is a scalar quantity measuring the rate at which the C bends or curves at the point $\mathbf{r}(t)$. This is given by
$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$
- Example. Consider a circle of radius r, parametrized by $\mathbf{r}(\theta) = r\cos(\theta)\mathbf{i} + r\sin(\theta)\mathbf{j}$. We compute the curvature to be $K = 1/r$. Observe that K is constant, with larger curvature the smaller r is.
- The previous example gives a geometric/intuitive way to think about curvature: at the given point $\mathbf{r}(t)$ on the curve C, draw a circle that best approximates C near $\mathbf{r}(t)$ (called an *osculating circle*). Then the curvature at $\mathbf{r}(t)$ is $1/r$, where r is the radius of the osculating circle.
- Let us add to what the textbook includes for your general awareness: if we take the cross product of the unit tangent and normal vectors, we get the *binormal vector* $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$. The vectors $\mathbf{T}(t)$, $\mathbf{N}(t)$, $\mathbf{B}(t)$ together span all of $\mathbb{R}^3$ and are called the *Frenet–Serret frame.* In addition to the curvature K, there is a quantity called the *torsion* that measures the extent to which the motion bends away from the plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$. From an initial starting frame, the curvature and torsion completely determine the path of motion of the particle. Thus we have a model of particle motion in $\mathbb{R}^3$. [Related personal anecdote: Some time ago, I watched the movie "Hidden Figures" at the recommendation of my parents. When watching math movies, I'm always curious how they incorporate math jargon. I remember (and looking now at the script confirmed) that the writers included the phrase "Frenet frame" to fill their need for impressive-sounding math jargon. So apparently this stuff gets used at NASA.]

- One essential point about the definition of curvature is that it is independent of the specific parametrization used. This suggests the idea of a standard parametrization of a curve: the *arc-length parametrization*. This is the parametrization for which $\|\mathbf{r}'(t)\| = 1$. For an arc length parametrization $\mathbf{r}(s)$, we have $K = \|T'(s)\|$.
- The *arc length* of a curve C with parametrization $\mathbf{r}(t)$, $a \leq t \leq b$, is $\int_a^b \|\mathbf{r}'(t)\|\, dt$. In fact, one can define the arc length function $s\colon [a,b] \to [0,\infty)$ by
$$s(t) = \int_a^t \|\mathbf{r}'(u)\|\, du = \int_a^t \sqrt{x'(u)^2 + y'(u)^2 + z'(u)^2}\, du.$$
If we're given a parametrization $\mathbf{r}(t)$ of a curve C, we can always find an arc-length parametrization of the same curve by solving for t in terms of s in the formula above. Note however that this might be computationally difficult.
- Example. Consider the curve $\mathbf{r}(t) = 3t\mathbf{i} - t\mathbf{j} + t^2\mathbf{k}$. Compute the tangential and normal components of acceleration.

  The answer is $a_{\mathbf{T}} = 4t/\sqrt{10+4t^2}$ and $a_{\mathbf{N}} = 2\sqrt{10}/\sqrt{10+4t^2}$.