MAT 203 LECTURE OUTLINE 9/27

- Given a function $z = f(x, y)$ and a point (x_0, y_0), we can find the tangent plane to the graph of $f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$. This is essentially the same thing as asking for the linearization of $f(x, y)$ at (x_0, y_0). Recall that the solution to this is
$$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0).$$
- We want to consider the same problem but for other surfaces. A surface does not need to be the graph of a function; a simple example is the sphere, given by the equation $x^2 + y^2 + z^2 = 1$. One way to think of a surface is as the level set of a function of three variables: the set of points satisfying $F(x, y, z) = 0$ for some differentiable function $F(x, y, z)$.
- The same ideas we discussed about the gradient for two-variable functions apply here. Namely, the gradient is a vector that is orthogonal to the level set $F(x, y, z) = 0$ at each point and that points in the direction of maximum increase for $F(x, y, z)$. The equation of the tangent plane at (x_0, y_0, z_0) is then
$$F_x(x, y, z)(x - x_0) + F_y(x, y, z)(y - y_0) + F_z(x, y, z)(z - z_0) = 0.$$
Observe how this is consistent with the equation in the first bullet point.
- Example. Consider the hyperboloid $z^2 - 2x^2 - 2y^2 = 12$. Find the tangent plane at $(1, -1, 4)$.
  This is given by $x - y - 2z + 6 = 0$.
- We begin section 13.8 on Extrema. We recall some definitions. Consider a function $f(x, y)$ defined on a planar region D.
  - The region D is *closed* if it contains all its boundary points.
  - The region D is *bounded* if it contained inside a ball of radius r for some $r > 0$.
  - The *maximum* (or *absolute maximum*) of $f(x, y)$ (if it exists) is the value M such that (i) $f(a, b) = M$ for some (a, b) in D, and (ii) $f(x, y) \leq M$ for all (x, y) in D.
  - The *minimum* (or *absolute minimum*) of $f(x, y)$ (if it exists) is the value m such that (i) $f(a, b) = m$ for some (a, b) in D, and (ii) $m \leq f(x, y)$ for all (x, y) in D.
- We also define *relative/local maxima* and *relative/local minima* to be values such that the same property holds if we replace D by **some disk** around the point (a, b).
- The Extreme Value Theorem states that **if** D is closed and bounded, **then** any continuous function f defined on D has a maximum and a minimum. That is, there is a point (a, b) in D such that $f(x, y) \leq f(a, b)$ for all points (x, y) in D, and a point (c, d) in D such that $f(c, d) \leq f(x, y)$ for all points (x, y) in D. If D is not closed or not bounded, then such a function f may or may not have a maximum or maximum. Also, note that a maximum or minimum may be attained by multiple points in D.
  Convince yourself intuitively that the Extreme Value Theorem is true, and that its conclusion may fail if D is not closed or not bounded.
- A *critical point* of f is a point (x, y) in the interior of a domain D where $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, or one of these partial derivatives does not exist. Here is the main principle of this section:
  Any relative minimum or maximum must occur at a boundary point or a critical point.
- One method to classify critical points is the "second partials test". Here is the statement:
  Suppose that $f(x, y)$ is a function with continuous second partial derivatives satisfying $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let
$$d = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2.$$
  (1) If $d > 0$ and $f_{xx}(a, b) > 0$, then f has a relative maximum at (a, b).
  (2) If $d > 0$ and $f_{xx}(a, b) < 0$, then f has a relative minimum at (a, b).
  (3) If $d < 0$, then f has a saddle point at (a, b).
  (4) If $d = 0$, the test is inconclusive.