MAT 203 LECTURE OUTLINE 9/29

- Today we are going to continue with the topic of finding extrema of a function. In applications, we can think of this as **optimization** of a function. We will start by doing a typical optimization problem.
- Example. Find the maximum volume of a box $[0, x] \times [0, y] \times [0, z]$ where we require the vertex (x, y, z) to lie on the plane $6x + 4y + 3z = 24$. That is, we want to maximize the function $f(x, y) = xyz = xy(24 - 6x - 4y)/3$. (The answer is 64/3.)
- Note: the textbook contains a section on "least squares". In the interest of time, we will not cover it in this course.
- We turn to another type of optimization problem: **constrained** optimization. This means that we do not optimize the target function over all values of a function, but just over those values satisfying a given constraint. For example, imagine we are constructing a warehouse. There are two variables we'd like to optimize: the size of the warehouse, and the cost of construction. Clearly, we cannot optimize both of these at the same time, since a larger building would generally cost more. What we can do is to choose a fixed size or a fixed cost, and then try to optimize the other variable *subject to that constraint.*
- There is a very nice method to solve constrained optimization problems called *Lagrange multipliers.* This will be the final topic of Chapter 13.
- It is easiest to explain the method and the idea behind it by using an example. We'll use the example from the textbook. Consider the ellipse $\dfrac{x^2}{3^2} + \dfrac{y^2}{4^2} = 1$.

  Problem: find the rectangle with vertices $(\pm x, \pm y)$ of largest area inscribed in this ellipse.

  Note that the area of this rectangle is $4xy$. Thus we are optimizing the function $f(x, y) = 4xy$ subject to the constraint $g(x, y) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$.

  At this point, take a look at Figure 13.79 in the textbook showing the contour lines of both $f(x, y)$ and $g(x, y)$. The key idea is that the any extrema must occur when the contour lines for f and g are tangent. Equivalently, this means that $\nabla f(x, y)$ and $\nabla g(x, y)$ must be scalar multiples of one another: $\nabla f(x, y) = \lambda \nabla g(x, y)$ for some scalar λ.

  We compute $\nabla f(x, y) = \langle 4y, 4x \rangle$ and $\nabla g(x, y) = \langle 2x/9, y/8 \rangle$. Thus (along with the original constraint) we have the system

$$\left\{ \begin{array}{rcl} 4y & = & \lambda(2x/9) \\ 4x & = & \lambda y/8 \\ \frac{x^2}{3^2} + \frac{y^2}{4^2} & = & 1 \end{array} \right. .$$

  This is a system of three equations in three variables, so one expects this system to have a unique solution. Typically these can be solved in an *ad hoc* way.

  For example, it can be solved in the following way. Rearrange the first equation as $\lambda = 18y/x$. Plug this into the second equation to get $4x = 18y^2/(8x)$, or equivalently $x^2 = 9y^2/16$. Now use the last equation to get $y = \pm 2\sqrt{2}$. This implies that $x = \pm 3/\sqrt{2}$.

  The way we posed this problem implies that x, y are positive (in general, you have to account for all cases). So we have $(x, y) = (2/\sqrt{2}, 2\sqrt{2})$, which gives a maximum area of 24.
- Here is a general statement of the method of Lagrange multipliers. Suppose we want to maximize or minimzie the function $f(x, y)$ subject to the constraint $g(x, y) = c$, where f, g have continuous first partial derivatives. Then we can do this by solving the system of equations

$$\left\{ \begin{array}{rcl} f_x(x, y) & = & \lambda g_x(x, y) \\ f_y(x, y) & = & \lambda g_y(x, y) \\ g(x, y) & = & c \end{array} \right.$$

  for the variables x, y, λ. Note that we have three equations in three variables, so we expect this equation to typically have a finite set of solutions.
- This method can be adapted to three or more variables using the relationship $\nabla f = \lambda \nabla g$.
- Note that there is some overlap between these methods. For example, the problem in the second bullet can be treated as a Lagrange multiplier problem.