MAT 203 LECTURE OUTLINE 9/8

- Class problem 1: identify the conic section determined by the equation $x^2 - 2y^2 + z^2 = 0$.
- We are now beginning Chapter 12 on vector-valued functions. These are functions whose input (*domain*) is a single variable and whose output (*codomain* or *range*) is a vector (or, equivalently, an element of $\mathbb{R}^n$ for some $n \geq 2$). Think of this chapter as a sort of warm-up for the main part of the course. In later chapters, we'll look at functions whose input/domain is multiple variables, which is the true "multivariable calculus".
- Recall that a vector-valued function has the form $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} = \langle f(t), g(t), h(t)\rangle$. (We'll state things for $\mathbb{R}^3$ for simplicity. It should be clear how to adapt everything to $\mathbb{R}^2$ or even $\mathbb{R}^n$.) It is conventional to use the variable t here to suggest time and so (x, y, z) can be used for spatial coordinates. The ideas of differentiation and integration extend to vector-valued functions in the obvious way.
- The notions of *limit* and *continuity* are defined component-wise:

$$\lim_{t\to a} \mathbf{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle,$$

  provided the limits on the left exist. Next, $\mathbf{r}$ is *continuous* at a if $\mathbf{r}(a) = \lim_{t\to a} \mathbf{r}(t)$.
- The derivative of a vector-valued function at t is defined as the *difference quotient*

$$\mathbf{r}'(t) = \lim_{\Delta t\to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$

  provided that this limit exists. In this case, we say that $\mathbf{r}$ is *differentiable* at t.
- It is straightforward to show that this definition agrees with differentiating each component function in the sense of single-variable calculus. That is, if $\mathbf{r}$ is differentiable at t, then

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

  Conversely, if $f'(t), g'(t), h'(t)$ all exist, then $\mathbf{r}$ is differentiable at t.
- Differentiation rules like the product rule and chain rule extend to vector-valued functions in the natural way:

$$\begin{aligned}
\frac{d}{dt}(c\,\mathbf{r}(t)) &= c\,\mathbf{r}'(t) \\
\frac{d}{dt}(\mathbf{r}(t) + \mathbf{u}(t)) &= \mathbf{r}'(t) + \mathbf{u}'(t) \\
\frac{d}{dt}(w(t)\mathbf{r}(t)) &= w(t)\mathbf{r}'(t) + w'(t)\mathbf{r}(t) \\
\frac{d}{dt}(\mathbf{r}(t)\cdot\mathbf{u}(t)) &= \mathbf{r}(t)\cdot\mathbf{u}'(t) + \mathbf{r}'(t)\cdot\mathbf{u}(t) \\
\frac{d}{dt}(\mathbf{r}(t)\times\mathbf{u}(t)) &= \mathbf{r}(t)\times\mathbf{u}'(t) + \mathbf{r}'(t)\times\mathbf{u}(t) \\
\frac{d}{dt}\mathbf{r}(w(t)) &= \mathbf{r}'(w(t))w'(t).
\end{aligned}$$

  These can all straightforward to check.
- Observe that the derivative $\mathbf{r}'$ is also a vector-valued function. In particular, we can take multiple derivatives of $\mathbf{r}$ just as in calculus of one variable. Geometrically, $\mathbf{r}'(t)$ represents the tangent vector of $\mathbf{r}$ at t. In many cases, $\mathbf{r}(t)$ represents the position of a particle traveling through space in time, and $\mathbf{r}'(t)$ then is the velocity of this particle. The second derivative $\mathbf{r}''(t)$ is the acceleration of this particle. Speed is the magnitude of velocity: $\|\mathbf{r}'(t)\|$.
- We can find an antiderivative for $\mathbf{r}$ by integrating each component function. This is called the *indefinite integral* of $\mathbf{r}$. Explicitly,

$$\int \mathbf{r}(t)\,dt = \left(\int f(t)\,dt\right)\mathbf{i} + \left(\int g(t)\,dt\right)\mathbf{j} + \left(\int h(t)\,dt\right)\mathbf{k}.$$

- The *definite integral* is defined similarly:
$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}.$$
As in calculus of one variable, one can take the definite integral of velocity from a to b to find the change in position of a particle between times a and b. Likewise, one can take the definite integral of acceleration from a to b to find the change in velocity of a particle between times a and b.
- Even if $\mathbf{r}$ is differentiable everywhere with continuous derivative, the curve traced by $\mathbf{r}$ may have corners or cusps (called *nodes*) at points t where $\mathbf{r}'(t) = 0$. (On the other hand, if $\|\mathbf{r}'(t)\| > 0$ for all t, then a continuously differentiable function traces out a smooth curve.)
- A historically important example is planetary motion. This is beyond the scope of this class, but look up *Kepler's laws* if you are interested. Here, the particle is a planet that moves in relation to a sun located at the origin based on the gravity of the sun. The motion is governed by the differential equation
$$\mathbf{r}'' = \frac{C}{\|\mathbf{r}\|^2}\left(\frac{-\mathbf{r}}{\|\mathbf{r}\|}\right),$$
where C is a constant, since gravity obeys an inverse square law. It can be shown that the trajectory of a planet is a conic section: an ellipse, parabola or hyperbola. Note that a planet moves with varying speed (faster when the planet is closer to the sun), so $\mathbf{r}$ is not the standard parametrization of an ellipse/parabola/hyperbola.
- A more manageable example is *projectile motion*. This is a model of particle motion in which gravity exerts a constant downward force. We ignore air resistance and other forces. Working in imperial units, we take the value $-32\mathbf{j}$ as the acceleration due to gravity: $\mathbf{r}''(t) = -32\mathbf{j}$. The motion of the particle is completely determined by its initial position $\mathbf{r}_0$ and initial velocity $\mathbf{v}_0$. Integrating $\mathbf{r}''$ twice, we have
$$\mathbf{r}(t) = -16t^2\mathbf{j} + \mathbf{v}_0 t + \mathbf{r}_0.$$
- Example. Consider an archer who shoots an arrow from an initial height of 4 ft. at a speed of 225 ft./s and initial angle θ over flat ground. How far does the arrow travel horizontally before hitting the ground?

  Applying the previous equation, we have
$$\mathbf{r}(t) = 225\cos(\theta)t\mathbf{i} + (-16t^2 + 225\sin(\theta)t + 4)\mathbf{j}.$$
The time when the arrow hits the ground is the solution to the quadratic equation
$$-16t^2 + 225\sin(\theta)t + 4 = 0.$$
From the quadratic equation, the answer is
$$225\cos(\theta)\left(\frac{225\sin(\theta) + \sqrt{225^2\sin^2(\theta) + 16^2}}{32}\right)\text{ ft.}$$
You might be curious about the farthest possible distance our archer can shoot. This occurs for $\theta = \pi/4$ and gives ≈ 1586.02 ft. In real life, we'd expect the actual distance to be somewhat less because of air resistance.
- Example. Circular motion. The position of a particle is given by the equation
$$\mathbf{r}(t) = \langle 3\sin(2t), 3\cos(2t)\rangle,$$
where $0 \le t \le \pi/2$. Sketch the trajectory of the particle, including the initial point and terminal point. Find the velocity, speed and acceleration.

  The answers are: $\mathbf{r}'(t) = \langle 6\cos(2t), -6\sin(2t)\rangle$, $\|\mathbf{r}'(t)\| = 6$ and $\mathbf{r}''(t) = \langle -12\sin(2t), -12\cos(2t)\rangle$. Observe how the acceleration is a scalar multiple of position, and both position and acceleration are orthogonal to velocity. This is the nature of circular motion at a constant speed.