Review: Green's theorem

Green's theorem

Let R be a simply connected region with piecewise smooth boundary C, oriented counterclockwise. Let $\mathbf{F} = \langle M, N \rangle$, where M, N have continuous first partial derivatives. Let $\mathbf{N}$ be the outward pointing normal vector. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R \text{curl}\, \mathbf{F}\, dA$$

and

$$\int_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_R \text{div}\, \mathbf{F}\, dA.$$

The divergence theorem: introduction

- Stoke's Theorem and the Divergence Theorem generalize the two versions of Green's theorem to three dimensions.
- Today we will look at the Divergence Theorem.
- Some terminology: a surface S is *closed* if it forms the complete boundary of a solid region in $\mathbb{R}^3$. All surfaces are assumed to be piecewise smooth.
- Any closed surface is orientable, and it is convention to chose the orientation given by outward-pointing unit normal vectors

The divergence theorem

Let Q be a region in $\mathbb{R}^3$ bounded by a closed surface S. Let $\mathbf{N}$ be the outward-pointing unit normal vector, and consider S as being oriented by $\mathbf{N}$. Let $\mathbf{F}$ be a vector field with continuous first partial derivatives in Q. Then

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iiint_Q \operatorname{div}\mathbf{F}\,dV.$$

- The notation $\oiint_S \mathbf{F}\cdot\mathbf{N}\,dS$ is also used.

Interpretation

- Physically, the integrals in the Divergence theorem may model the flow of a fluid through a permeable membrane, with **F** the velocity field of the fluid.
- A point where div **F** > 0 is called a source
- A point where div **F** < 0 is called a sink
- A point where div **F** $= 0$ is called incompressible