Practice Exam 2

1. • $\vec{r}'(t) = \langle e^t, -\pi \sin(\pi t), \pi \cos(\pi t) \rangle$

$\vec{r}'(1) = \langle e, 0, -\pi \rangle$

$\vec{r}(1) = \langle e, -1, 0 \rangle$

tangent line: $\vec{T}(t) = \langle e + te, -1, -\pi t \rangle$

• length $= \int_0^3 \|\vec{v}'(t)\| \, dt = \int_0^3 2\sqrt{5}\, t e^{t^2} dt = \sqrt{5}\, e^{t^2} \Big|_0^3 = \sqrt{5}\,(e^9 - 1)$

$\vec{v}'(t) = \langle 2te^{t^2}, 4te^{t^2} \rangle$

$\|\vec{v}'(t)\| = \sqrt{4t^2 e^{2t^2} + 16t^2 e^{2t^2}} = 2te^{t^2}\sqrt{1+4} = 2\sqrt{5}\, t e^{t^2}$

2. $T(x,y) = 200 - 3x^2 + y^2$

$\nabla T(x,y) = \langle -6x, 2y \rangle$

$\frac{dx}{dt} = -6x \qquad \frac{dy}{dt} = 2y$

$\Longrightarrow \frac{1}{-6x}\, dx = dt \qquad \frac{1}{2y}\, dy = dt$

$-\frac{1}{6}\ln(x) = t + c_1 \qquad \frac{1}{2}\ln(y) = t + c_2$

$\ln(x) = -6t + c_1 \qquad \ln(y) = 2t + c_2$

$x = c_1 e^{-6t} \qquad y = c_2 e^{2t}$

$x(0) = c_1 = 2 \qquad y(0) = c_2 = 1$

$\vec{w}(t) = \langle 2e^{-6t}, e^{2t} \rangle$

3. $f(x,y,z) = x^2 + y^2 + z^2 \qquad g(x,y,z) = x + y + z = 30$

$\nabla f = \lambda \nabla g$: $\left\{ \begin{array}{l} 2x = \lambda \\ 2y = \lambda \\ 2z = \lambda \\ x + y + z = 30 \end{array} \right\} \Longrightarrow x = y = z = 10$

Detailed justification for 3:

[To see this is a minimum, note that $0 \le x \le 30$,
$0 \le y \le 30 - x$,
$z = 30 - x - y$
is the domain of f.
So we must compare $f(10,10,10) = 30$
with the boundary of the domain.
By symmetry we may assume
$z = 0$ and $y = 30 - x$.
Then $f(x,y,z) = x^2 + (30-x)^2 =: g(x)$
$\Longrightarrow g'(x) = 2x + 2(30-x)(-1) = 4x - 60 =$
$x = 15$
$g(0) = 30^2 = 900$, $g(30) = 900$, $g(15) = 45$
All of these are bigger than 300.]

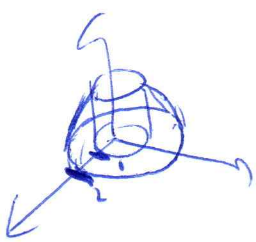
4. $F(x,y,z) = z^2 - 2x^2 - 2y^2 = 12$

$\nabla F(x,y,z) = \langle -4x, -4y, 2z \rangle$

$\nabla F(1,-1,4) = \langle -4, 4, 8 \rangle$

$-4(x-1) + 4(y+1) + 8(z-4) = 0$

5. [Assume $0 \le z$.]



$$\int_0^{2\pi}\int_1^2\int_0^{4-r^2} r\,dz\,dr\,d\theta = 2\pi\int_1^2 (4-r^2)\,r\,dr$$

$$= 2\pi\left[-\frac{1}{2}\cdot\frac{1}{2}(4-r^2)^2\right]_1^2 = \frac{\pi}{2}(4-1)^2 = \boxed{\frac{9\pi}{2}}$$

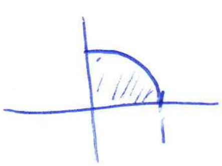
6. $g(x,y) = 2x^2 + y^2 = 1$

$\nabla f = \lambda \nabla g \Longrightarrow \begin{cases} 2 = \lambda(4x) \\ 1 = \lambda(2y) \\ 2x^2 + y^2 = 1 \end{cases} \Longrightarrow \begin{array}{l} x = \frac{1}{2\lambda} \\ y = \frac{1}{2\lambda} \end{array}$

$\frac{2}{4\lambda^2} + \frac{1}{4\lambda^2} = 1 \longrightarrow \lambda^2 = \frac{3}{4}$

$\lambda = \pm\frac{\sqrt{3}}{2}$

7.



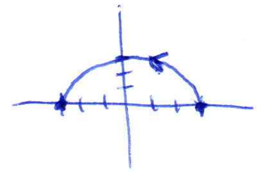
$$m = \int_0^{\pi/2}\int_0^1 (r^2)\,r\,dr\,d\theta = \frac{\pi}{2}\int_0^1 r^3\,dr = \frac{\pi}{2}\left.\frac{r^4}{4}\right|_0^1 = \boxed{\frac{\pi}{8}}$$

$$\bar{x} = \frac{1}{m}\int_0^{\pi/2}\int_0^1 (r\cos\theta)\,r^2 r\,dr\,d\theta = \frac{1}{m}\int_0^{\pi/2}\cos(\theta)\,d\theta\int_0^1 r^4\,dr = \frac{1}{m}\left(\sin(\theta)\Big|_0^{\pi/2}\right)\left(\frac{1}{5}r^5\Big|_0^1\right)$$

$$= \frac{1}{m}(1)\left(\frac{1}{5}\right) = \boxed{\frac{8}{5\pi}}$$

$(\bar{x}, \bar{y}) = \left(\frac{8}{5\pi}, \frac{8}{5\pi}\right)$

8. $\vec{F} = \langle 2x-3y, -3x-y+4 \rangle$



Find potential:

$f(x,y) = x^2 - 3xy + g(y)$

$f_y(x,y) = -3x + g'(y) = -3x - y + 4$

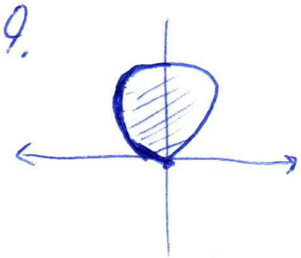
$\Longrightarrow \quad g'(y) = -y + 4$

$g(y) = -\frac{1}{2}y^2 + 4y$

$f(x,y) = x^2 - 3xy - \frac{1}{2}y^2 + 4y$

$f(-3,0) - f(3,0) = 9 - 9 = 0$

9.



$\text{area} = \int_C x\,dy = \int_0^{\pi} \sin(2t)\cos(t)\,dt = \int_0 2\sin(t)\cos^2(t)\,dt$

$(dy = \cos(t)\,dt)$

$= -\frac{2}{3}\cos^3(t)\Big|_0^{\pi}$

$= -\frac{2}{3}(-1-1) = \boxed{\frac{4}{3}}$