Exam 5 - Fall 2018

3. (a) $F(x,y,z) = e^{xyz} + 3x^2$

$\nabla F(x,y,z) = \langle yze^{xyz} + 6x,\ xze^{xyz},\ xye^{xyz} \rangle$

$\nabla F(1,0,3) = \langle 0+6,\ 3(1),\ 0 \rangle = \langle 6,3,0 \rangle$

$$\vec{n} = \frac{\langle 6,3,0\rangle}{\|\langle 6,3,0\rangle\|} = \frac{\langle 6,3,0\rangle}{\sqrt{36+9}} = \left\langle \frac{6}{3\sqrt{5}}, \frac{3}{3\sqrt{5}}, 0 \right\rangle = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0 \right\rangle$$

(b) $\frac{2}{\sqrt{5}}(x-1) + \frac{1}{\sqrt{5}}(y-0) + 0(z-3) = 0$

$\Longrightarrow$ $2x - 2 + y = 0$

4. $F(x,y) = \ln(x^2+y^2+2y+2)$, $\quad D = \{(x,y) : x^2+y^2 < 16\}$

$$\frac{\partial F}{\partial x} = \frac{2x}{x^2+y^2+2y+2} = 0 \Longrightarrow x = 0$$

$$\frac{\partial F}{\partial y} = \frac{2y+2}{x^2+y^2+2y+2} = 0 \Longrightarrow y = -1$$

$(x,y) = (0,-1)$ is the only critical point. (Note that $x^2+y^2+2y+2 = x^2+(y+1)^2+1 \geq 1$, so the denominator is positive.)

$$\frac{\partial^2 F}{\partial x^2} = \frac{(x^2+y^2+2y+2)(2) - (2x)^2}{(x^2+y^2+2y+2)^2}$$

$$\frac{\partial^2 F}{\partial y^2} = \frac{(x^2+y^2+2y+2)(2) - (2y+2)^2}{(x^2+y^2+2y+2)^2}$$

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{-(2x)(2y+2)}{(x^2+y^2+2y+2)^2}$$

$$d = \begin{vmatrix} \frac{(1-2+2)(2)}{(1-2+2)^2} & 0 \\ 0 & \frac{(1-2+2)(2)-(-2+2)^2}{(1-2+2)^2} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$$

$(0,-1)$ is a local minimum, since $d = 4 > 0$ and $F_{xx} > 0$

5. $F(x,y) = x^2 + 2y,\quad g(x,y) = x^2 + y^2 - 6y$

$\nabla F = \lambda \nabla g \implies \begin{cases} 2x = \lambda(2x) \\ 2 = \lambda(2y-6) \\ x^2+y^2-6y = 0 \end{cases}$

$\implies (x=0)$ or $(\lambda = 1)$

$x=0 \implies y^2 - 6y = 0 \implies y=0$ or $y=6$

$\lambda = 1 \implies 2 = 2y-6 \implies y=4 \implies x^2+16-24=0$

$\implies x^2 - 8 = 0 \implies x = \pm\sqrt{8}$

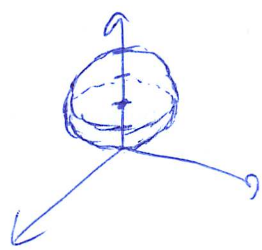
Critical points: $(0,0), (0,6), (\sqrt{8},4), (-\sqrt{8},4)$

$F(0,0) = 0 \quad F(\sqrt{8},4) = 8+8 = 16$

$F(0,6) = 12 \quad F(-\sqrt{8},4) = 8+8=16$

Maximum: 16 at $(\pm\sqrt{8},4)$ minimum: 0 at $(0,0)$

6. $$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{4\cos(\phi)} \frac{\rho\cos(\phi)e^{\rho^2}}{\cancel{\rho^2}}\cancel{\rho^2}\sin(\phi)\,d\rho\,d\phi\,d\theta$$



$$= \int_0^{2\pi}\int_0^{\pi/2}\left(\frac{1}{2}e^{\rho^2}\Big|_0^{4\cos\phi}\right)\cos(\phi)\sin(\phi)\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\frac{1}{2}\left(e^{16\cos^2(\phi)} - 1\right)\cos(\phi)\sin(\phi)\,d\phi\,d\theta$$

$x^2+y^2+z^2-4z+4=4$

$\implies \rho^2 = 4z = 4\rho\cos(\phi)$

$\implies \rho = 4\cos(\phi)$

$$= \int_0^{2\pi}\frac{1}{2}\left[\frac{-1}{32}e^{16\cos^2(\phi)} - \frac{\sin^2(\phi)}{2}\right]_0^{\pi/2}d\theta$$

$$= \int_0^{2\pi}\left[\frac{-1}{64} - \frac{1}{4} + \frac{1}{64}e^{16}\right]d\theta = 2\pi\left(\frac{e^{16}}{64} - \frac{17}{64}\right) = \pi\left(\frac{e^{16}}{32} - \frac{17}{32}\right)$$

8. $\vec{r}(t) = \langle t^2 + 5, t \rangle$

$\vec{r}\,'(t) = \langle 2t, 1 \rangle \qquad \|\vec{r}\,'(t)\| = \sqrt{4t^2+1}$

$x = t^2+5 \qquad y = t$

$$\int_C 10x\sqrt{1+4y^2}\,ds = \int_0^1 10(t^2+5)(\sqrt{1+4t^2})\sqrt{4t^2+1}\,dt$$

$$= \int_0^1 10(t^2+5)(4t^2+1)\,dt = 10\int_0^1 4t^4 + 20t^2 + t^2 + 5\,dt$$

$$= 10\left[\frac{4t^5}{5} + \frac{21}{3}t^3 + 5t\right]_0^1 = 10\left[\frac{4}{5} + \frac{21}{3} + 5\right]$$

$$= 10\left[\frac{12}{15} + \frac{105}{15} + \frac{75}{15}\right] = 10 \cdot \frac{192}{15} = \frac{2}{3} \cdot 192 = 2 \cdot 64 = \boxed{128}$$