Exam 6 – Spring 2018

1. $f(x,y) = x^2 - xy - y^2 - 3x - y$

$f_x = 2x - y - 3 = 0 \qquad f_y = -x - 2y - 1 = 0$

$$\Rightarrow \begin{cases} 2x - y = 3 \\ x + 2y = -1 \end{cases}$$

$$\underline{4x - 2y = 6}$$

$$5x = 5 \Rightarrow x = 1 \rightarrow y = -1$$

Critical point: $(1,-1)$ (saddle point)

$f_{xx} = 2 \qquad f_{xy} = -1$

$f_{yy} = -2$

$$d = \begin{vmatrix} 2 & -1 \\ -1 & -2 \end{vmatrix} = -4 - 1 = -5 < 0 \Rightarrow \text{saddle point}$$

2. $\|\vec{r}\,'(t)\| = \| -\sin(t)\,\vec{\imath} + \cos(t)\,\vec{\jmath}\,\| = 1$. $x = \cos(t)$, $y = \sin(t)$

$$M = \int_0^{\pi/2} \big(\cos(t) + \sin(t)\big)\, dt = \sin(t) + -\cos(t)\Big|_0^{\pi/2} = 1 - 0 - 0 + 1 = \boxed{2}$$

3. (1) $\vec{F} = \langle M, N\rangle = \langle -e^x \sin(y), -e^x \cos(y)\rangle$

$$\text{curl}\,\vec{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -e^x \cos(y) + e^x \cos(y) = 0$$

$\cdot\vec{F}$ is defined on all $\mathbb{R}^2$, so $\vec{F}$ is <u>conservative</u>.

(2) $f(x,y) = \int -e^x \sin(y)\, dx = -e^x \sin(y) + h(y)$ $\qquad f(x,y) = -e^x \sin(y)$

$f_y = -e^x \cos(y) + h'(y) \Rightarrow h(y) = 0$

(3) $f(1, 2\pi) - f(0,\pi) = -e^1 \sin(2\pi) + e^0 \sin(\pi) = \boxed{0}$

4. (1) $F(x,y,z) = xy - z = 0$

$\nabla F = \langle y, x, -1\rangle \qquad \nabla F(2,3,6) = \langle 3, 2, -1\rangle$

$$3(x-2) + 2(y-3) + -1(z-6) = 0$$

$$3x - 6 + 2y - 6 - z + 6 = 0$$

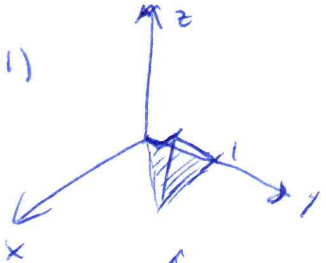
$$\boxed{3x + 2y - z = 6}$$

(2)

$$\iint_D \sqrt{1 + y^2 + x^2}\, dA = \int_0^{2\pi}\int_0^1 \sqrt{1+r^2}\, r\, dr\, d\theta = \frac{1}{2}\frac{2}{3}(1+r^2)^{3/2}\Big|_0^1 (2\pi) = \left(\frac{1}{3}2^{3/2} - \frac{1}{3}\right)(2\pi)$$

$$= \boxed{\frac{2\pi}{3}\left(2^{3/2} - 1\right)}$$

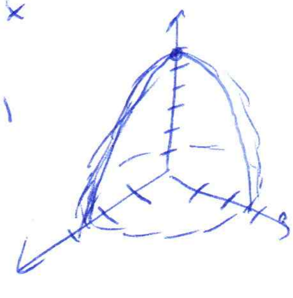
5. (1)



$$\int_0^1\int_0^y\int_0^{xy} 1\,dz\,dx\,dy = \int_0^1\int_0^y xy\,dx\,dy = \int_0^1 \frac{1}{2}x^2y\Big|_0^y\,dy$$

$$= \int_0^1 \frac{1}{2}y^3\,dy = \frac{1}{8}y^4\Big|_0^1 = \boxed{\frac{1}{8}}$$

(2)

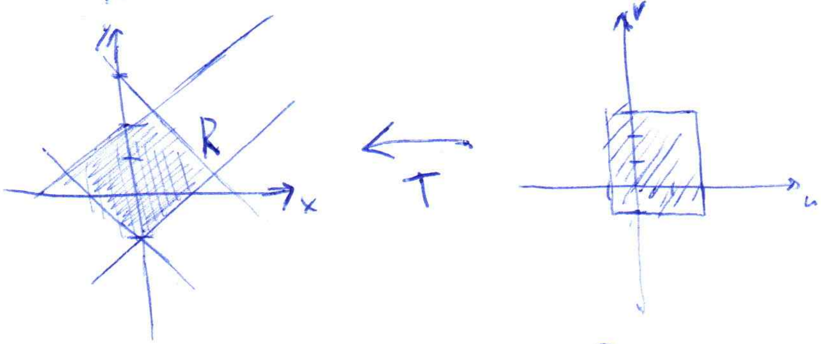


$$\int_0^{2\pi}\int_0^{\sqrt{6}}\int_0^{6-r^2} (1)\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{6}} (6-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} 3r^2 - \frac{1}{4}r^4\Big|_0^{\sqrt{6}}\,d\theta = 2\pi\left(3\cdot 6 - \frac{1}{4}\cdot 36\right)$$

$$= 2\pi(18-9) = \boxed{18\pi}$$

6. $\iint_R (2x+2y)\,dA$



$$(x,y) = T(u,v) = \left(\frac{v-u}{2}, \frac{v+u}{2}\right)$$

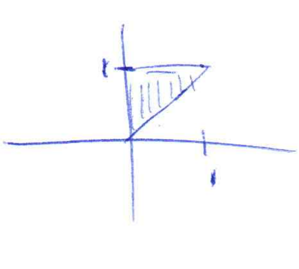
$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = \left|-1/4 - 1/4\right| = \left|-1/2\right| = 1/2$$

$$y+x = \frac{v+u+v-u}{2} = \frac{2v}{2} = v \Longrightarrow -1 \le v \le 3$$

$$y-x = \frac{v+u-v+u}{2} = \frac{2u}{2} = u \Longrightarrow -1 \le u \le 2$$

$$\int_{-1}^{2}\int_{-1}^{3} (2v)\frac{1}{2}\,dv\,du = \int_{-1}^{2} \frac{1}{2}v^2\Big|_{-1}^{3}\,du = \frac{1}{2}\int_{-1}^{2} 9-1\,du = \frac{8(3)}{2} = \boxed{\frac{24}{2}} = \boxed{12}$$

7. (1) $\int_0^1\int_x^1 e^{y^2}\,dy\,dx = \int_0^1\int_0^y e^{y^2}\,dx\,dy = \int_0^1 e^{y^2}y\,dx = \frac{1}{2}e^{y^2}\Big|_0^1 = \boxed{\frac{1}{2}(e-1)}$



(2) $\int_0^1\int_0^1 x^n y^n\,dx\,dy = \int_0^1 \frac{x^{n+1}}{n+1}y^n\Big|_0^1\,dy = \int_0^1 \frac{1}{n+1}y^n\,dy$

$$= \frac{1}{(n+1)^2}y^{n+1}\Big|_0^1 = \frac{1}{(n+1)^2} \qquad \lim_{n\to\infty}\frac{1}{(n+1)^2} = \boxed{0}$$