# MAT 132

## Practice Midterm II Solutions.

This is a closed notes/ closed book/ electronics off exam.

Please write legibly and show your work.

Each problem is worth 20 points.

| Full Name: | | | | | | |
|---|---|---|---|---|---|---|
| Problem | 1 | 2 | 3 | 4 | 5 | Total |
| Grade | | | | | | |

**Problem 1.** Find the volume of a solid whose base is a circle of radius 1, and whose horizontal cross-sections are squares.

**Solution 1.** At distance $x$ from the center of the circle the square cross section has side length $2\sqrt{1-x^2}$ and area $4(1-x^2)$. Thus the volume is

$$V = \int_{-1}^{1} 4(1-x^2)dx = 4x - \frac{4}{3}x^3\Big]_{-1}^{1} = \frac{16}{3}.$$

**Problem 2.** Let $R$ be the region

$$R = \{(x, y) : 0 \le x \le \pi, 0 \le y \le \sin x\}.$$

a. Find the center of mass of $R$.
b. Let $S$ be the solid found by rotating $R$ about the $y$ axis. Find the volume of $S$.
c. Let $T$ be the solid found by rotating $R$ about the $x$ axis. Find the volume of $T$.

**Solution 2.**

a. Let the center of mass be $(\overline{x}, \overline{y})$. By symmetry $\overline{x} = \frac{\pi}{2}$. The area of the figure is

$$A = \int_0^\pi \sin x dx = 2.$$

The first moment in the $y$ direction is

$$M_y = \frac{1}{2}\int_0^\pi (\sin x)^2 dx = \frac{\pi}{4}.$$

Hence $\overline{y} = \frac{M_y}{A} = \frac{\pi}{8}$.

b. By the method of cylindrical shells, the volume of rotation is

$$\mathrm{Vol}(S) = \int_0^\pi 2\pi x \sin x dx = 2\pi M_x.$$

Since $M_x = A\overline{x} = \pi$, $\mathrm{Vol}(S) = 2\pi^2$.

c. By the method of cross-sectional slices,

$$\mathrm{Vol}(T) = \int_0^\pi \pi(\sin x)^2 dx = 2\pi M_y = \frac{\pi^2}{2}.$$

**Problem 3.** Find the length of a curve given by the parametric equations $x(t) = e^t \cos t$, $y(t) = e^t \sin t$, $0 \le t \le \pi$. Find the average $x$ and $y$ coordinates over this interval.

**Solution 3.** We have

$$x'(t) = e^t \cos t - e^t \sin t, \qquad y'(t) = e^t \sin t + e^t \cos t.$$

Thus

$$x'(t)^2 + y'(t)^2 = e^{2t}\left[(\cos t - \sin t)^2 + (\sin t + \cos t)^2\right] = 2e^{2t}.$$

Thus the arc length is

$$\int_0^\pi \sqrt{x'(t)^2 + y'(t)^2}dt = \sqrt{2}\int_0^\pi e^t dt = \sqrt{2}(e^\pi - 1).$$

The average $x$ value is

$$\overline{x} = \frac{1}{\pi}\int_0^\pi e^t \cos t dt.$$

Integrate by parts with $u = e^t$, $du = e^t dt$, $dv = \cos t dt$, $v = \sin t$. Thus

$$\int e^t \cos t dt = e^t \sin t - \int e^t \sin t dt.$$

Integrate by parts again, now with $u = e^t$, $du = e^t dt$, $dv = \sin t dt$, $v = -\cos t$. Thus

$$\int e^t \cos t dt = e^t \sin t + e^t \cos t - \int e^t \cos t dt$$

so

$$\int e^t \cos t dt = \frac{1}{2}\left[e^t \sin t + e^t \cos t\right] + C.$$

Thus

$$\overline{x} = \frac{1}{\pi}\left[\frac{1}{2}(e^t \sin t + e^t \cos t)\right]_0^\pi = \frac{1}{2\pi}(-1 - e^\pi).$$

The average $y$ value is

$$\overline{y} = \frac{1}{\pi}\int_0^\pi e^t \sin t dt.$$

Integrate by parts with $u = e^t$, $du = e^t dt$, $dv = \sin t dt$, $v = -\cos t$ to find

$$\int e^t \sin t dt = -e^t \cos t + \int e^t \cos t dt = \frac{1}{2}\left[e^t \sin t - e^t \cos t\right] + C.$$

Thus $\overline{y} = \frac{1}{2\pi}(e^\pi + 1)$.

**Problem 4.** A 5kg mass is attached to a spring with spring constant $K = 20kg/s^2$. The spring is stretched to 10 cm and then released, after which it exhibits simple harmonic motion with displacement from equilibrium in cm

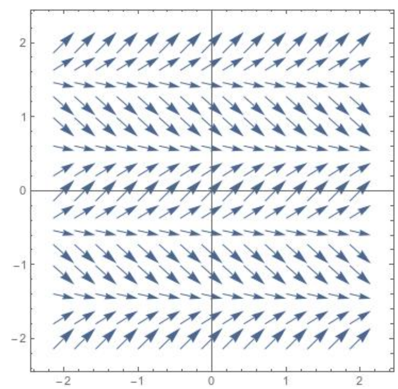
$$x(t) = 10\cos(2t).$$

Find the work done in Joules on the mass by the spring between $t = 0$ and $t = \frac{\pi}{2}$.

**Solution 4.** At time $t = \frac{\pi}{2}$ the mass is at position $-10$ and over the interval the mass moves from position 10 to position -10. While moving from 10 to 0, the force from the spring is in the direction of motion, while from 0 to -10 it opposes the motion. By symmetry, the work done is 0, since the magnitude of the force at $x$ and $-x$ is equal.

**Problem 5.**

a. Given the initial value problem $y' = 2(x + y - 1)$, $y(0) = 1$, use Euler's method with step $h = \frac{1}{2}$ to estimate $y(2)$.
b. Find the equilibria of the time homogeneous differential equation $y' = \cos(\pi y)$. An equilibrium $y = a$ is called stable if there is a small interval $I = [a - \delta, a + \delta]$ such that if $y(0) \in I$ then $y(t) \to a$ as $t \to \infty$. Which equilibria are stable?




**Solution 5.**

a.

| $x$ | $y$ | $y'$ |
|---|---|---|
| 0 | 1 | 0 |
| $\frac{1}{2}$ | 1 | 1 |
| 1 | $\frac{3}{2}$ | 3 |
| $\frac{3}{2}$ | 3 | 7 |
| 2 | $\frac{13}{2}$ | |

Euler's method thus estimates $y(2) \approx \frac{13}{2}$.

b. Equilibria occur where $y' = 0$ for all $t$, and hence, where $\cos(\pi y) = 0$. This occurs when $y$ differs from an integer by $\frac{1}{2}$.

An equilibrium $a$ is stable if there is an interval $[a - \delta, a + \delta]$ such that $\cos(\pi y) > 0$ for $a - \delta \leq y < a$ and $\cos(\pi y) < 0$ for $a < y \leq a + \delta$ and unstable when the opposite inequalities are true. This is because, under the stable condition, the solution is moving towards $a$ with positive derivative at any given time, and crosses any separating threshold from $a$ in finite time. This may be proven rigorously using the mean value theorem. The stable equilibria are $\frac{1}{2}$ more than twice an integer, i.e. ..., $-\frac{3}{2}$, $\frac{1}{2}$, $\frac{5}{2}$, $\frac{9}{2}$,....