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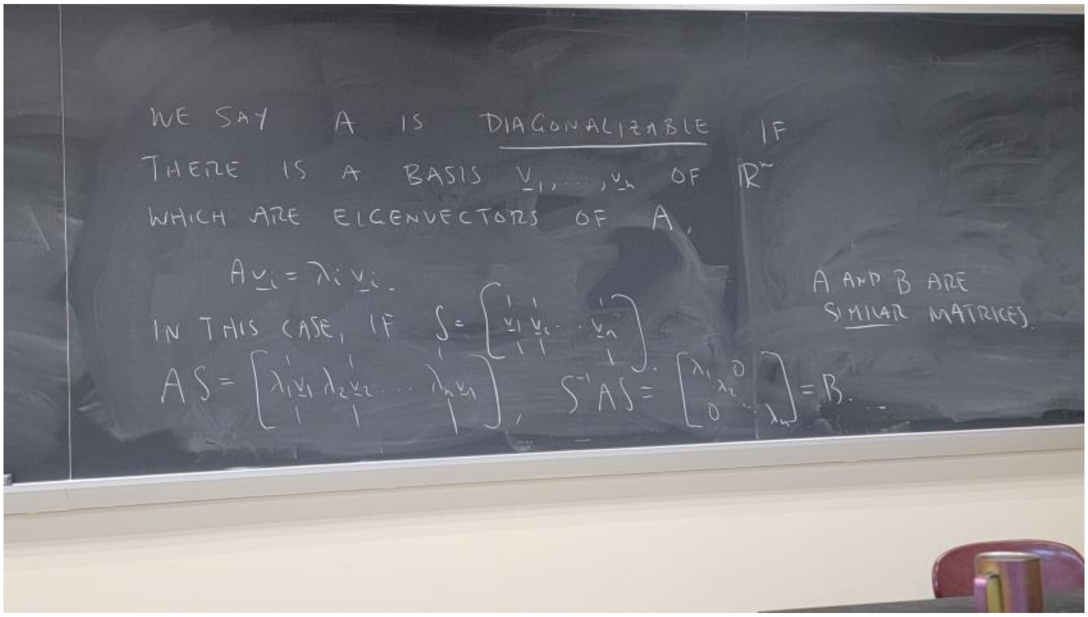
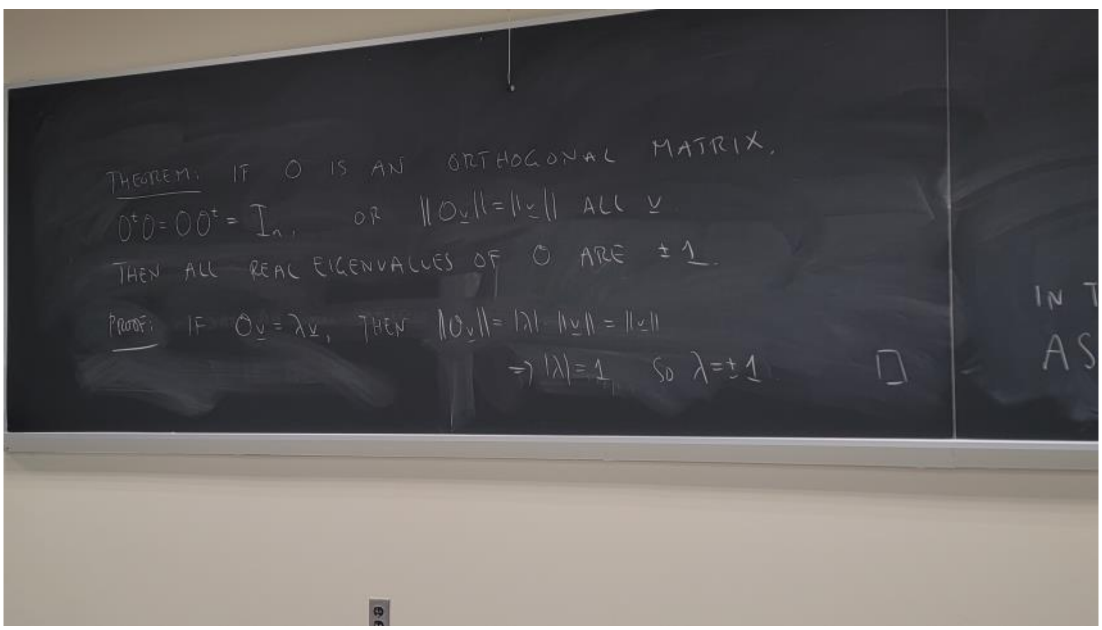
Saturday, April 8, 2023 5:32 PM





DISCRETE DYNAMICAL SYSTEMS:

A (LINEAR) DYNAMICAL SYSTEM IS A SEQUENCE OF STATES $\underline{x}_0, \underline{x}_1, \underline{x}_2, \underline{x}_3, \dots$

(E.G. POPULATION OF WOLVES, DEER IN A HABITAT, THE LIST OF STOCK PRICES OF COMPANIES IN SOME SECTOR OF THE ECONOMY, THE MOMENTA OF SOME PARTICLES, ETC.)

THE SYSTEM IS <u>LINEAR</u> IF THERE IS MATRIX A SO THAT

$$x_{t+1} = A x_t.$$

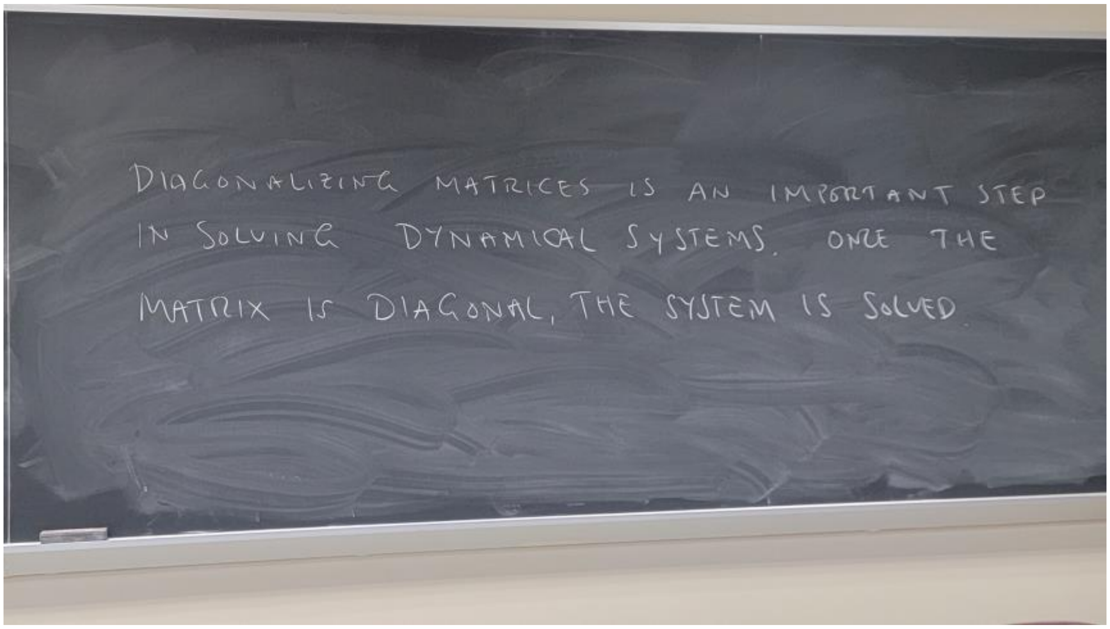
IN THIS CASE $x_t = A^t x_0$.

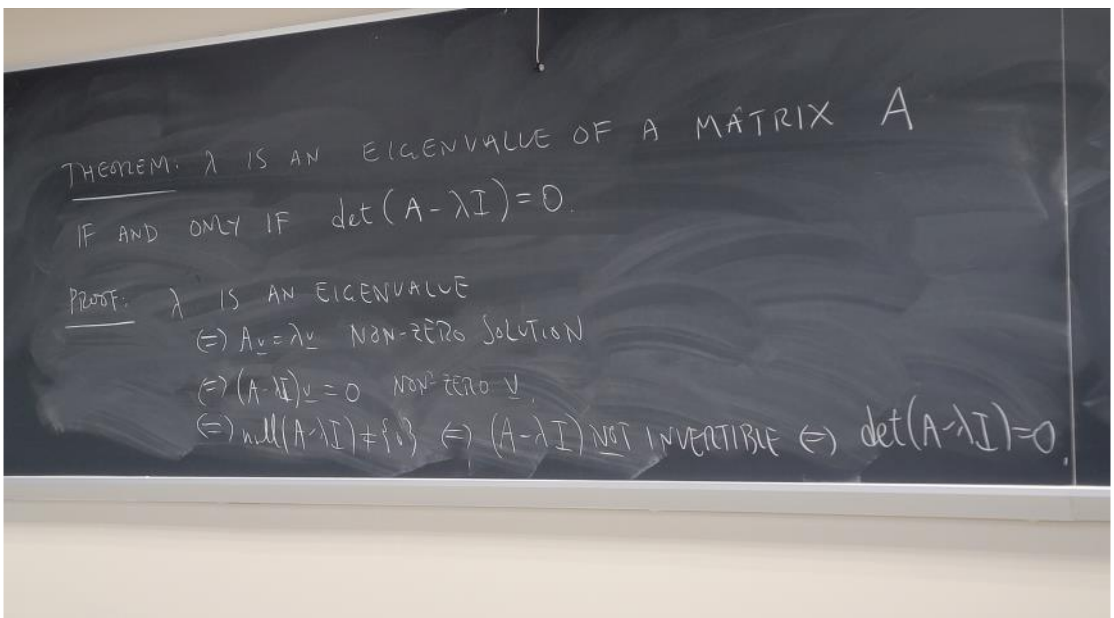
$$x_1 = Ax_0, \quad x_2 = Ax_1 = A^2 x_0, \quad \dots$$

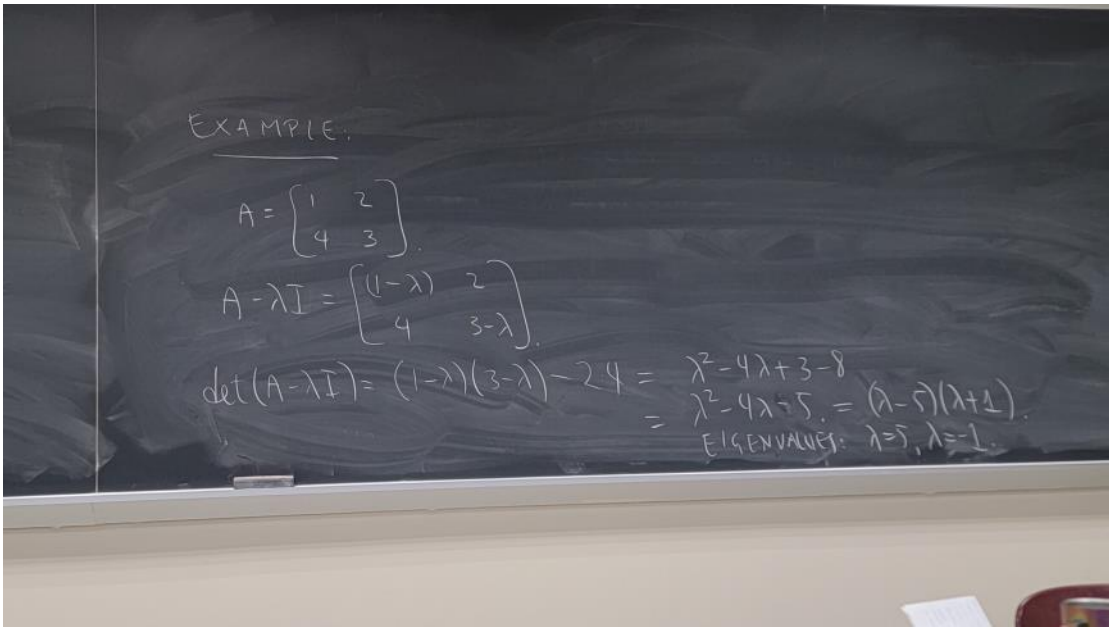
IF A IS <u>DIAGONAL</u>, $A = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$, $A^t = \begin{bmatrix} \lambda_1^t & & 0 \\ & \lambda_2^t & \\ 0 & & \ddots \; \lambda_n^t \end{bmatrix}$.

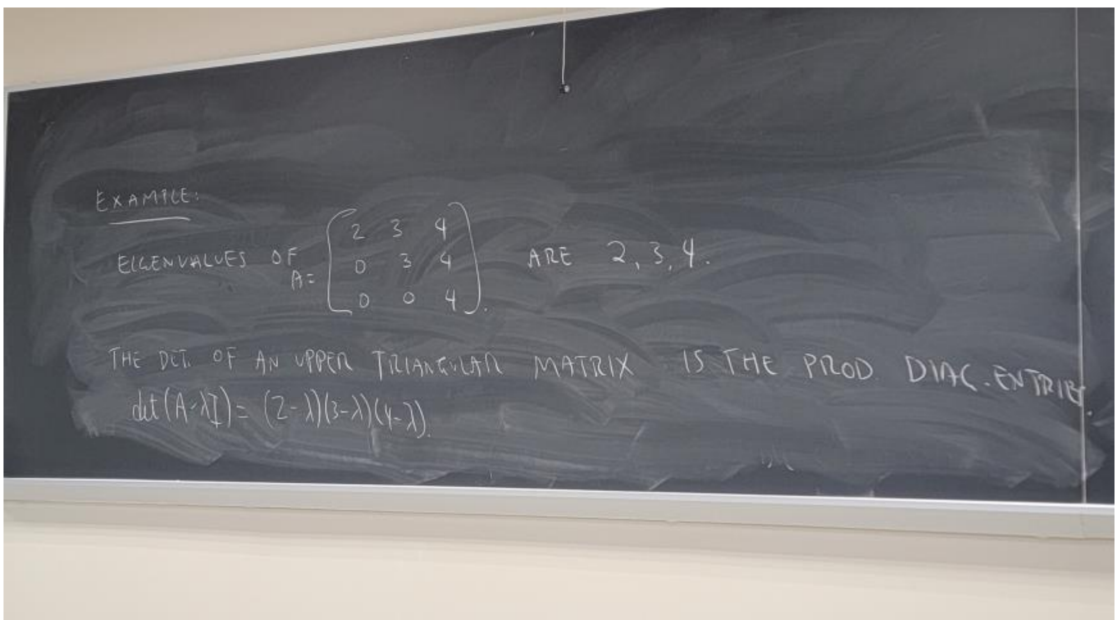
IF $A = S^{-1} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix} S$

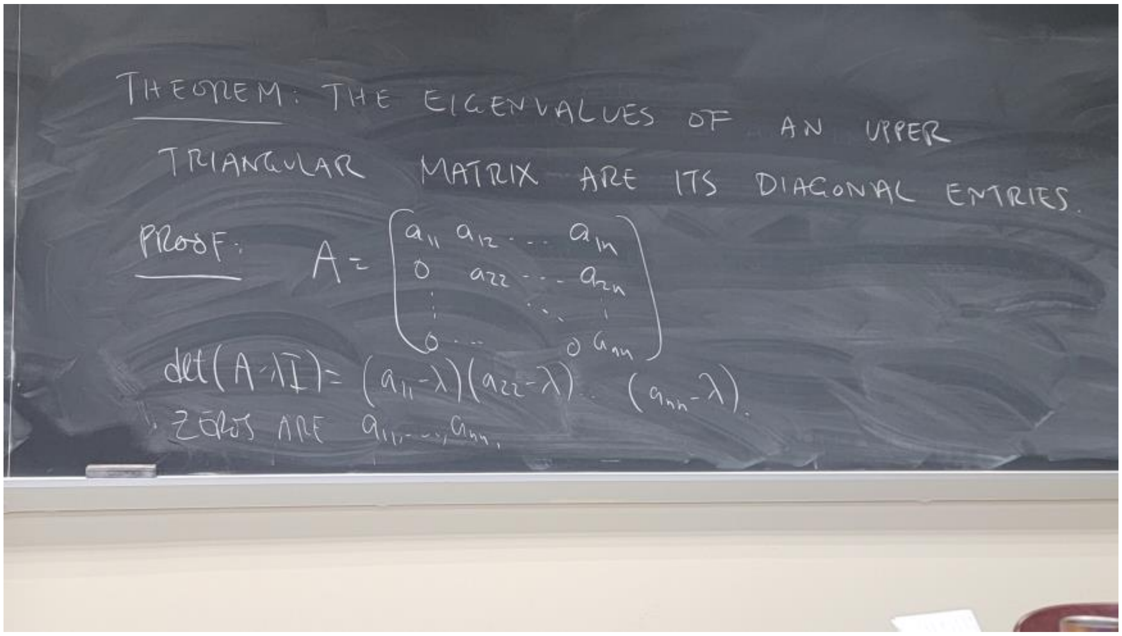
$A^t = S^{-1} \begin{bmatrix} \lambda_1^t & & 0 \\ & \lambda_2^t & \\ 0 & & \ddots \; \lambda_n^t \end{bmatrix} S$

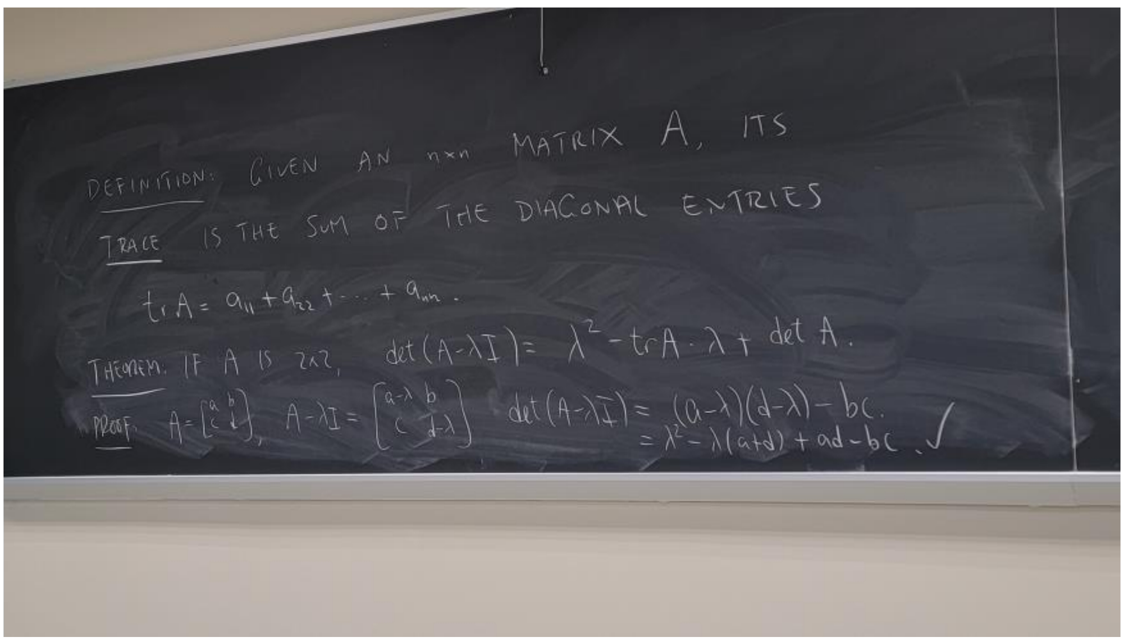

DIAGONALIZING MATRICES IS AN IMPORTANT STEP
IN SOLVING DYNAMICAL SYSTEMS. ONCE THE
MATRIX IS DIAGONAL, THE SYSTEM IS SOLVED.


THEOREM: λ IS AN EIGENVALUE OF A MATRIX A
IF AND ONLY IF $\det(A-\lambda I)=0$.
PROOF: λ IS AN EIGENVALUE
$\Leftrightarrow A\underline{v}=\lambda\underline{v}$ NON-ZERO SOLUTION
$\Leftrightarrow (A-\lambda I)\underline{v}=0$ NON-ZERO $\underline{v}$.
$\Leftrightarrow \mathrm{null}(A-\lambda I)\neq\{0\} \Leftrightarrow (A-\lambda I)$ NOT INVERTIBLE $\Leftrightarrow \det(A-\lambda I)=0$.


EXAMPLE:
A = [1 2; 4 3].
A − λI = [(1−λ) 2; 4 3−λ].
det(A−λI) = (1−λ)(3−λ) − 2·4 = λ² − 4λ + 3 − 8
= λ² − 4λ − 5. = (λ−5)(λ+1).
EIGENVALUES: λ=5, λ=−1.


EXAMPLE:
EIGENVALUES OF A= [2 3 4; 0 3 4; 0 0 4]. ARE 2, 3, 4.
THE DET. OF AN UPPER TRIANGULAR MATRIX IS THE PROD. DIAG. ENTRIES.
det(A−λI) = (2−λ)(3−λ)(4−λ).


THEOREM: THE EIGENVALUES OF AN UPPER TRIANGULAR MATRIX ARE ITS DIAGONAL ENTRIES.
PROOF: $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix}$
$\det(A-\lambda I) = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$.
$\therefore$ ZEROS ARE $a_{11}, \dots, a_{nn}$.


DEFINITION: GIVEN AN $n \times n$ MATRIX A, ITS TRACE IS THE SUM OF THE DIAGONAL ENTRIES
$\operatorname{tr} A = a_{11} + a_{22} + \cdots + a_{nn}$.
THEOREM. IF A IS 2×2, $\det(A-\lambda I) = \lambda^2 - \operatorname{tr}A \cdot \lambda + \det A$.
PROOF: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A-\lambda I = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$ $\det(A-\lambda I) = (a-\lambda)(d-\lambda) - bc$
$= \lambda^2 - \lambda(a+d) + ad - bc$. ✓

DEFINITION: IF A IS AN $n \times n$ MATRIX,

$$f_A(\lambda) = \det(A - \lambda I)$$

IS CALLED THE <u>CHARACTERISTIC POLYNOMIAL</u>.

THIS IS A POLYNOMIAL OF DEGREE n,

$$f_A(\lambda) = (-\lambda)^n + (\operatorname{tr} A)(-\lambda)^{n-1} + \dots + \det(A)$$

PROOF:

$$f_A(\lambda) = \det \begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{pmatrix}$$

WE GAVE A FORMULA FOR THE DETERMINANT OF A MATRIX B

$$\det B = \sum_{\sigma \text{ PERMUTATION}} \operatorname{sgn}(\sigma)\, B_{1\sigma(1)} \cdots B_{n\sigma(n)}$$

EACH TERM IN THIS SUM IS A POLY IN λ, DEG $\leq n$

$\Rightarrow f_A(\lambda)$ IS A POLY IN λ, DEG $\leq n$.

TO GET A DEGREE n TERM,
WE MUST SELECT EACH DIAGONAL ENTRY

$$(a_{11}-\lambda)\cdots(a_{nn}-\lambda) = \underbrace{(-\lambda)^n}_{\text{TOP ORDER TERM.}} + (a_{11}+\cdots+a_{nn})(-\lambda)^{n-1} + \cdots$$

IF YOU SELECT ALL BUT ONE DIAGONAL ENTRY, MUST SELECT ALL
SO THE REMAINING TERMS HAVE DEG $\leq n-2$.

$$\Rightarrow f_A(\lambda) = (-\lambda)^n + (a_{11}+\cdots+a_{nn})(-\lambda)^{n-1} + \text{DEG} \leq n-2$$

TO GET THE CONSTANT TERM

$$f_A(0) = \text{CONSTANT TERM} = \det(A).$$

$\square$

DEFINITION: THE ALGEBRAIC MULTIPLICITY OF AN EIGENVALUE λ_0 IS THE DEGREE OF THE ROOT λ_0 IN THE CHARACTERISTIC POLYNOMIAL.

FACTOR: $f_A(\lambda) = (-1)^n(\lambda-\lambda_1)^{d_1}\cdots(\lambda-\lambda_k)^{d_k}$.

(OVER THE COMPLEX NUMBERS THE FUNDAMENTAL THEOREM OF ALGEBRA SAYS THAT EVERY POLYNOMIAL FACTORS INTO LINEAR FACTORS.)

THE ALGEBRAIC MULTIPLICITIES ARE $d_1, \dots, d_k$.

EXAMPLE:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}.$$

RANK$(A) = 1$.

$\dim \operatorname{null}(A) = 2$.

$\dim \operatorname{null}(A - 3I) = 1$

$\dim \operatorname{null}(A) = 2$.

CHAR POLY OF A

$$(3-\lambda)\cdot(-\lambda)^2 = f_A(\lambda).$$

$$\det\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} = (-\lambda)^3 + 3(-\lambda)^2 + \overset{?}{(0)}\cdot(-\lambda) + 0$$

$$\boxed{\lambda^2(3-\lambda)}$$

det A.

$$= (1-\lambda)\det\begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} - 1\det\begin{pmatrix} 1 & 1 \\ 1 & 1-\lambda \end{pmatrix} + 1\det\begin{bmatrix} 1 & 1-\lambda \\ 1 & 1 \end{bmatrix}.$$

$$= (1-\lambda)\cdot\left[(1-\lambda)^2 - 1\right] - \left[(1-\lambda) - 1\right] + \left[1-(1-\lambda)\right]$$

$$(1-\lambda)\left[\lambda^2 - 2\lambda\right] + \lambda + \lambda.$$

THE λ TERM IS 0.

THEOREM: AN $n \times n$ MATRIX HAS AT MOST n REAL EIGENVALUES COUNTED WITH ALGEBRAIC MULTIPLICITY.

IF n IS ODD, IT HAS AT LEAST 1.

PROOF: ANY EIGENVALUE IS A ROOT OF THE CHAR. POLY, $\leq n$ ROOTS

NOT NEC. ALL REAL.

A POLY THAT IS ODD HAS A REAL ROOT (RULE OF SIGN).

EXAMPLE: FOR A 3×3 MATRIX, THE POSSIBILITIES FOR THE REAL EIGENVALUES ARE

1. λ_1 ALG. MULTIP. 1.
2. λ_1 ALG. MULT. 2, λ_2 ALG. MULT 1.
3. λ_1 MULT. 1, λ_2 MULT 1, λ_3 MULT. 1.

PROOF: $f_A(\lambda)$ FACTORS AS ONE OF

$-\lambda^3 + \operatorname{tr} A\, \lambda^2 + b\,\lambda + \det A$.

1. $-(\lambda - \lambda_1)$ DEG 2 IRRED QUAD POLY.
2. $-(\lambda - \lambda_1)^2 (\lambda - \lambda_2)$.
3. $-(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$.

THEOREM: GIVEN AN $n \times n$ MATRIX A WITH EIGENVALUES $\lambda_1, \ldots, \lambda_n$

$$\det A = \lambda_1 \cdot \ldots \cdot \lambda_n.$$

$$\operatorname{tr} A = \lambda_1 + \ldots + \lambda_n.$$

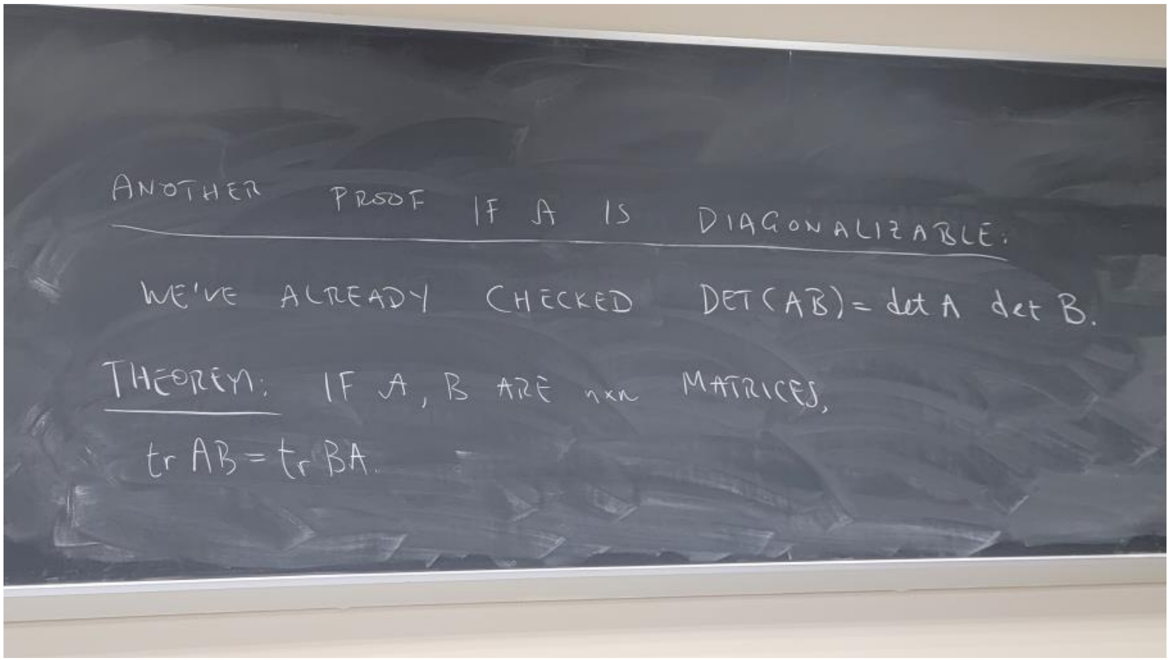
PROOF: $f_A(\lambda) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n).$

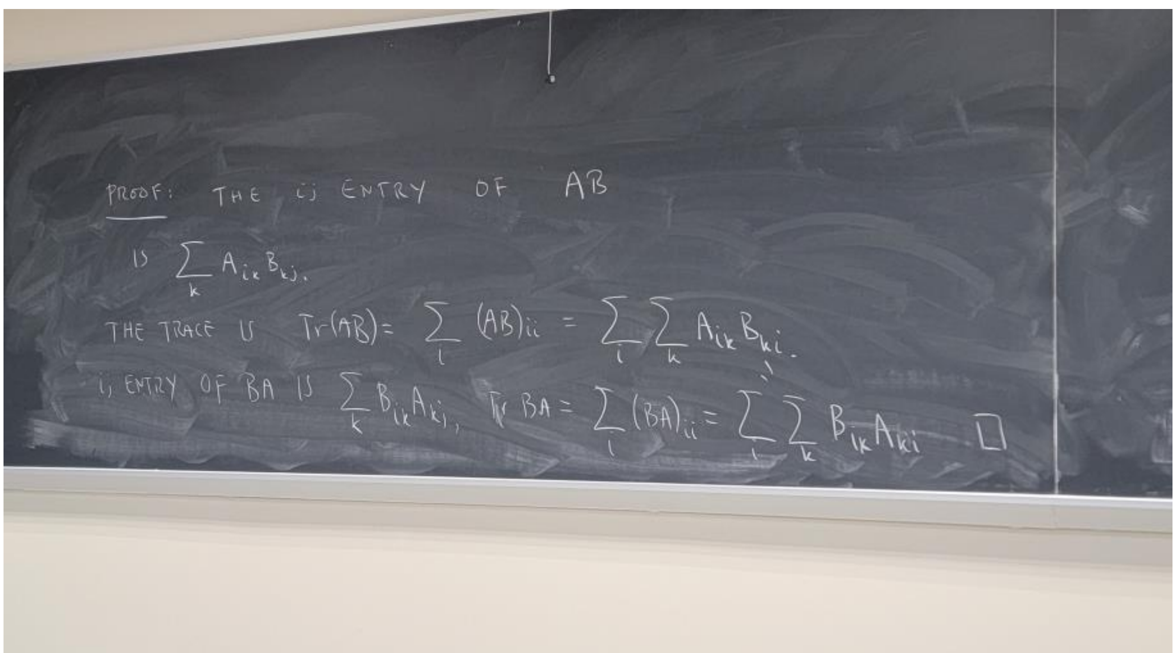
WE KNOW $f_A(\lambda) = (-\lambda)^n + (\operatorname{tr} A)(-\lambda)^{n-1} + \ldots + \det A.$

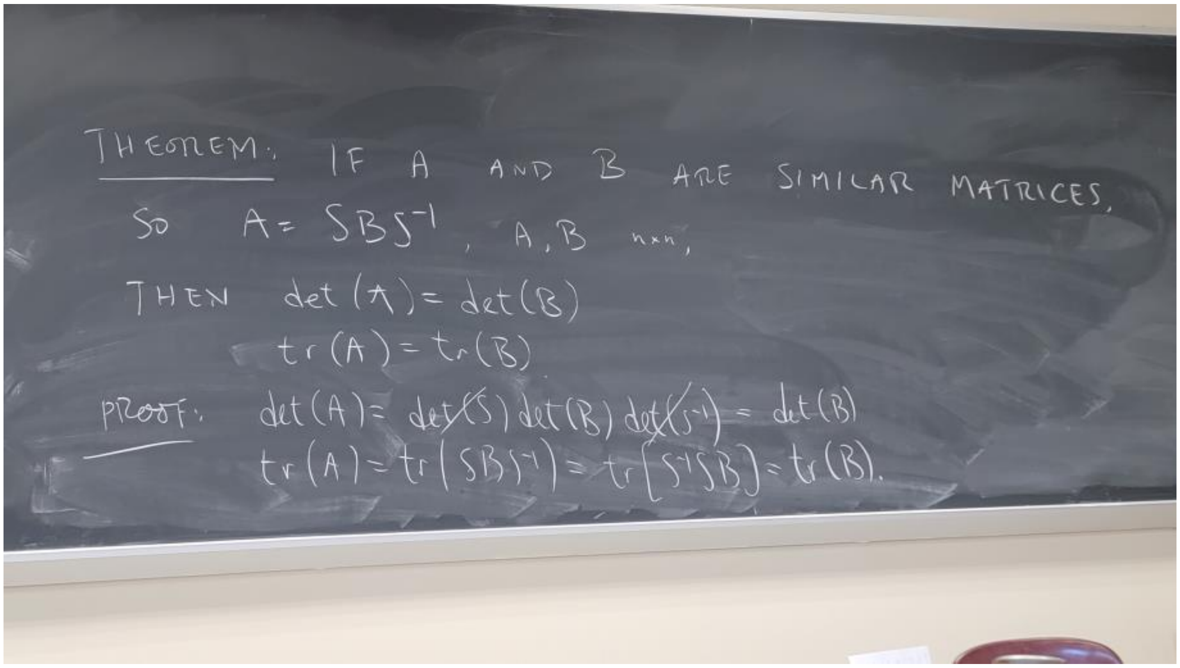
THE CONSTANT TERM IN $f_A(\lambda)$ IS $f_A(0) = \det A$
$= \lambda_1 \ldots \lambda_n.$

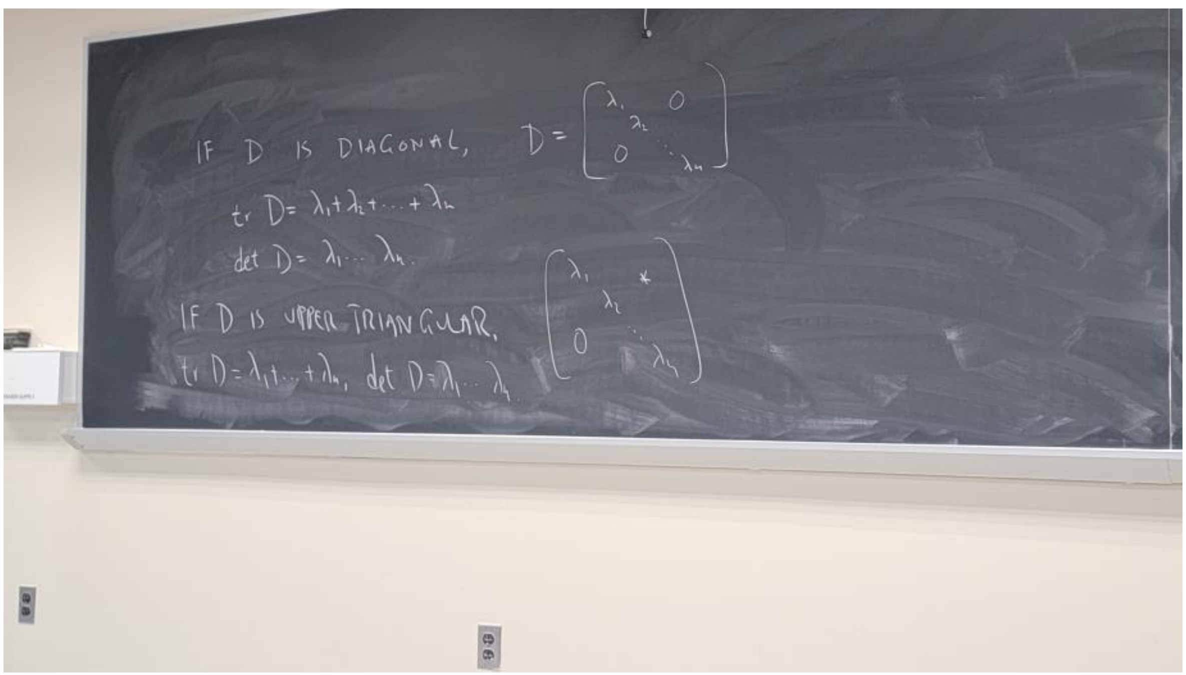
THE DEG. $n-1$ TERM IN $f_A(\lambda)$ IS $(-1)^{n-1} \operatorname{tr} A = (-1)^{n+1}(\lambda_1 + \ldots + \lambda_n).$

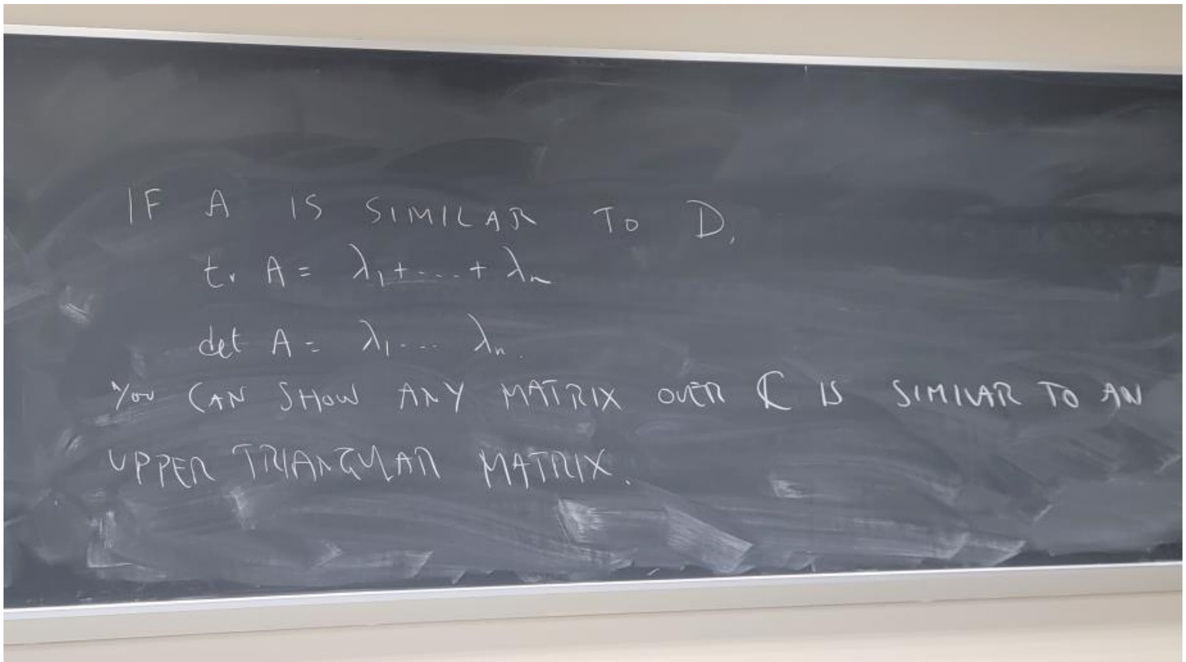
$\operatorname{tr} A = \lambda_1 + \ldots + \lambda_n.$

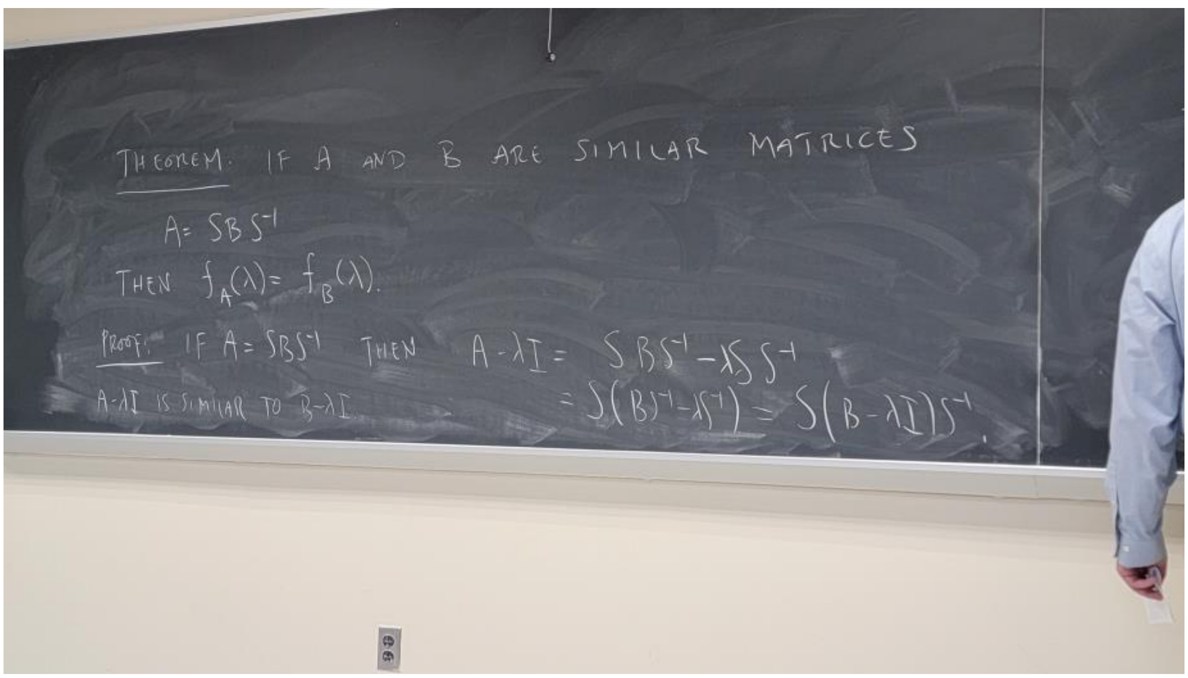

ANOTHER PROOF IF A IS DIAGONALIZABLE:
WE'VE ALREADY CHECKED DET(AB) = det A det B.
THEOREM: IF A, B ARE n×n MATRICES,
tr AB = tr BA.

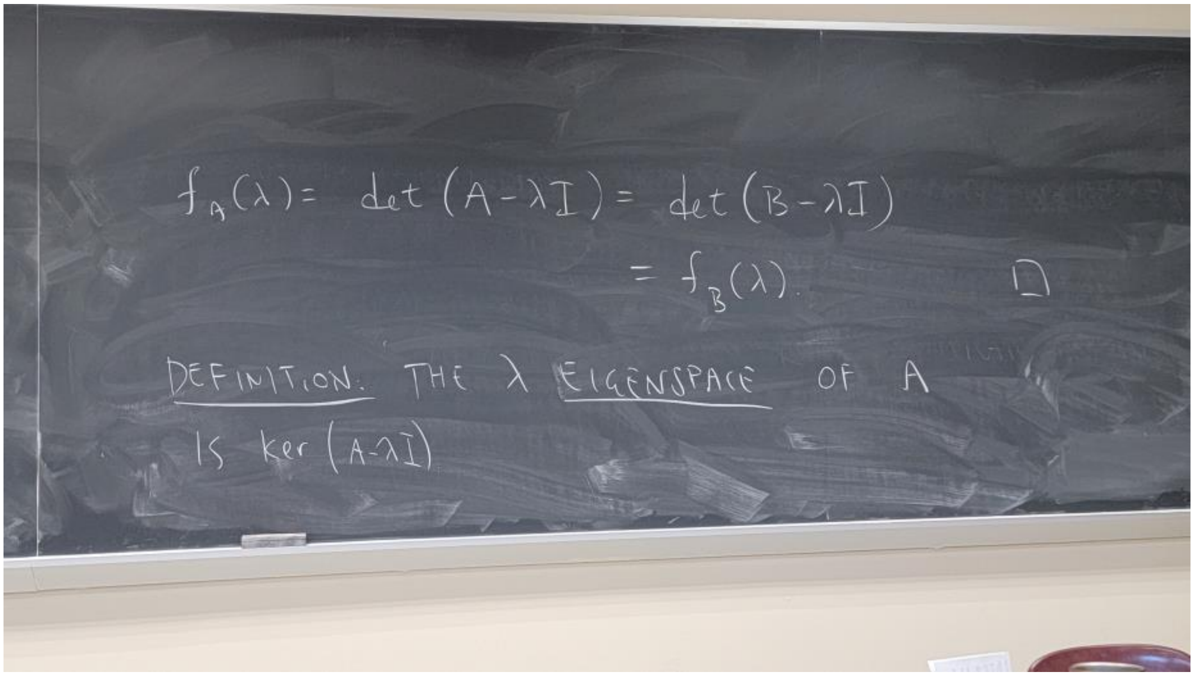

PROOF: THE ij ENTRY OF AB
IS $\sum_k A_{ik}B_{kj}$.
THE TRACE IS $\mathrm{Tr}(AB) = \sum_i (AB)_{ii} = \sum_i \sum_k A_{ik}B_{ki}$.
ij ENTRY OF BA IS $\sum_k B_{ik}A_{kj}$, $\mathrm{Tr}\, BA = \sum_i (BA)_{ii} = \sum_i \sum_k B_{ik}A_{ki}$ $\square$

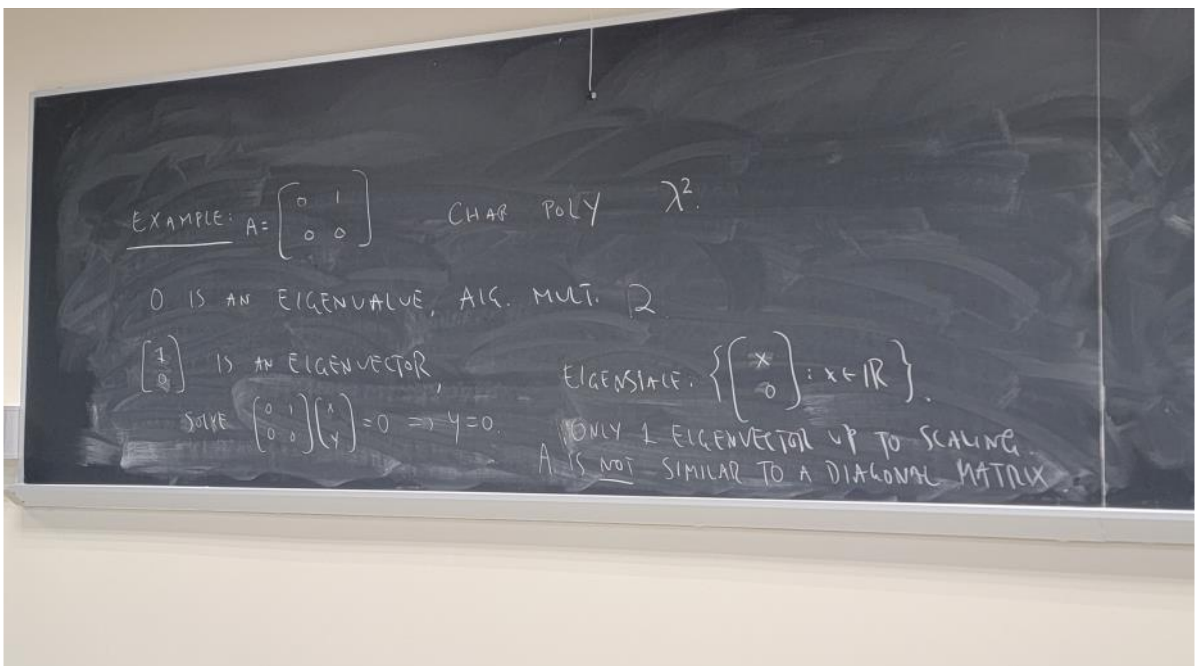

THEOREM: IF A AND B ARE SIMILAR MATRICES,
SO $A = SBS^{-1}$, A, B $n \times n$,
THEN $\det(A) = \det(B)$
$tr(A) = tr(B)$
PROOF: $\det(A) = \cancel{\det(S)} \det(B) \cancel{\det(S^{-1})} = \det(B)$
$tr(A) = tr(SBS^{-1}) = tr[S^{-1}SB] = tr(B)$.


IF D IS DIAGONAL, $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$
$tr\, D = \lambda_1 + \lambda_2 + \dots + \lambda_n$
$\det D = \lambda_1 \cdots \lambda_n$.
IF D IS UPPER TRIANGULAR, $\begin{bmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$
$tr\, D = \lambda_1 + \dots + \lambda_n$, $\det D = \lambda_1 \cdots \lambda_n$.


IF A IS SIMILAR TO D,
$\operatorname{tr} A = \lambda_1 + \dots + \lambda_n$
$\det A = \lambda_1 \cdots \lambda_n$.
YOU CAN SHOW ANY MATRIX OVER $\mathbb{C}$ IS SIMILAR TO AN
UPPER TRIANGULAR MATRIX.


THEOREM: IF A AND B ARE SIMILAR MATRICES
$A = SBS^{-1}$
THEN $f_A(\lambda) = f_B(\lambda)$.
PROOF: IF $A = SBS^{-1}$ THEN $A - \lambda I = SBS^{-1} - \lambda S S^{-1}$
$= S(BS^{-1} - \lambda S^{-1}) = S(B - \lambda I)S^{-1}$.
$A - \lambda I$ IS SIMILAR TO $B - \lambda I$.


$f_A(\lambda) = \det(A - \lambda I) = \det(B - \lambda I)$
$= f_B(\lambda)$. □
DEFINITION: THE λ EIGENSPACE OF A
IS $\ker(A - \lambda I)$


EXAMPLE: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ CHAR POLY λ^2.
0 IS AN EIGENVALUE, ALG. MULT. 2
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ IS AN EIGENVECTOR,
EIGENSPACE: $\left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\}$.
SOLVE $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow y = 0$.
ONLY 1 EIGENVECTOR UP TO SCALING.
A IS NOT SIMILAR TO A DIAGONAL MATRIX.

EXAMPLE: $T\underline{x} = A\underline{x}$ ORTHOGONAL PROJECTION TO A 2 DIMENSIONAL PLANE IN $\mathbb{R}^3$.

SAY $\underline{u}_1, \underline{u}_2$ SPAN THE PLANE, AND ARE ORTHONORMAL.

$$T\underline{x} = (\underline{x}\cdot\underline{u}_1)\underline{u}_1 + (\underline{x}\cdot\underline{u}_2)\underline{u}_2 = (\underline{u}_1^t\cdot\underline{x})\underline{u}_1 + (\underline{u}_2^t\cdot\underline{x})\underline{u}_2 = (\underline{u}_1\underline{u}_1^t) + (\underline{u}_2\underline{u}_2^t)\cdot\underline{x}.$$

HERE $\underline{u}_1$ IS 3×1

$\underline{u}_1^t = 1\times 3$

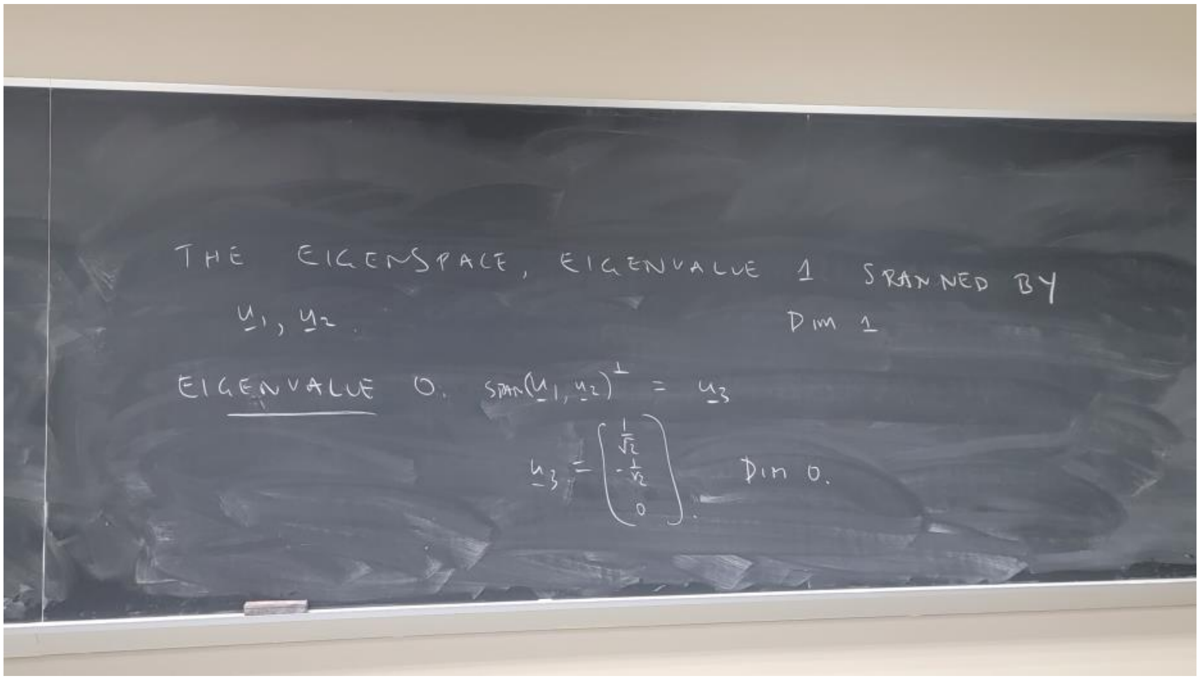
$\underline{u}_1\underline{u}_1^t$ IS 3×3.

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Ex. $\underline{u}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$, $\underline{u}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\underline{u}_1\underline{u}_1^t = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\underline{u}_2\underline{u}_2^t = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$


THE EIGENSPACE, EIGENVALUE 1 SPANNED BY
$\underline{u}_1, \underline{u}_2$.
Dim 1
EIGENVALUE 0. $\text{span}(\underline{u}_1, \underline{u}_2)^{\perp} = \underline{u}_3$
$\underline{u}_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$.
Dim 0.