SPRING 2020: MAT 320 MIDTERM 2 SOLUTIONS

The purpose of this midterm exam is to prove a version of the existence and uniqueness theorem for ordinary differential equations. Solve the problems independently without outside aids. Your solutions should be submitted on Blackboard as for the weekly homework assignments.

Problem 1. (20 pts) Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Hint: Fix $x_0 \in [a, b]$, let

$$M_0 = \sup |f(x)|, \qquad M_1 = \sup |f'(x)|$$

for $a \leq x \leq x_0$. For any such x,

$$|f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0.$$

Thus $M_0 = 0$ if $A(x_0 - a) < 1$.

Solution. Suppose for contradiction that $f(x) \neq 0$ for some $x \in [a, b]$. Let $\alpha = \inf\{x \in [a, b] : f(x) \neq 0\}$. Since f is continuous, $f(\alpha) = 0$. Let

$$M_0 = \sup_{\alpha \leq x \leq x_0} |f(x)|, \qquad M_1 = \sup_{\alpha \leq x \leq x_0} |f'(x)|$$

where $\alpha \leq x_0 < \alpha + \frac{1}{A}$. By the condition on f', $M_1 \leq AM_0$. By the Mean Value Theorem, for $y \in (\alpha, x_0]$, there is $c \in (\alpha, y)$ such that

$$f(y) = f(y) - f(\alpha) = f'(c)(y - \alpha)$$

and hence, $M_0 \leq M_1(x_0 - \alpha) \leq AM_0(x_0 - \alpha)$. Since $A(x_0 - \alpha) < 1$, it follows that $M_0 = 0$, a contradiction.

Problem 2. (30 pts) Let ϕ be a real function defined on the rectangle R in the plane, given by $a \leq x \leq b$, $\alpha \leq y \leq \beta$. A *solution* of the initial-value problem

$$y' = \phi(x, y), \qquad y(a) = c \qquad (\alpha \leq c \leq \beta)$$

is, by definition, a differentiable function f on $[a, b]$ such that $f(a) = c$, $\alpha \leq f(x) \leq \beta$, and

$$f'(x) = \phi(x, f(x)) \qquad (a \leq x \leq b).$$

Prove that such a problem has at most one solution if there is a constant A such that

$$|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$$

whenever $(x, y_1), (x, y_2) \in R$.

Solution. Let f_1, f_2 be two solutions of the initial value problem and let $g = f_1 - f_2$. Then $g(a) = 0$ and $g'(x) = f_1'(x) - f_2'(x) = \phi(x, f_1(x)) - \phi(x, f_2(x))$ so that

$$|g'(x)| \le A|g(x)|.$$

It follows from the previous problem that $g = 0$, and hence $f_1 = f_2$.

The following proof of the existence theorem for ODE's uses several concepts for functions of two variables which are generalizations of concepts for functions of one variable.

Definition 1. *A function $f : [a, b] \times [c, d] \to \mathbb{R}$ is continuous at (x_0, y_0) if, for any $\epsilon > 0$ there exists $\delta > 0$ such that, if $\|(x - x_0, y - y_0)\|_2 < \delta$ then $|f(x, y) - f(x_0, y_0)| < \epsilon$. A function f is uniformly continuous if, for any $\epsilon > 0$ there exists $\delta > 0$ such that, whenever $\|(x_1 - x_2, y_1 - y_2)\|_2 < \delta$, then $|f(x_1, y_1) - f(x_2, y_2)| < \epsilon$.*

You may use the fact that a continuous function on $[a, b] \times [c, d]$ is uniformly continuous. For a proof of this fact, see the taped version of Lecture 18.

Problem 3. (50 pts) Suppose ϕ is a continuous bounded real function in the strip defined by $0 \le x \le 1$, $-\infty < y < \infty$. Prove that the initial-value problem

$$y' = \phi(x, y), \qquad y(0) = c$$

has a solution. It may help to follow the following steps.

a. For each n, let $x_i = \frac{i}{n}$, $i = 0, 1, 2, ..., n$ and define a piecewise linear continuous function $f_n(t)$ by

$$f_n'(t) = \phi(x_i, f_n(x_i)), \qquad \text{if } x_i < t < x_{i+1}.$$

Define $\Delta_n(t) = f_n'(t) - \phi(t, f_n(t))$ except at $t = x_i$, where we define $\Delta_n(x_i) = 0$. Explain why

$$f_n(x) = c + \int_0^x [\phi(t, f_n(t)) + \Delta_n(t)]dt.$$

b. Let M be such that $|\phi(x, y)| \le M$. Prove that $|f_n'(x)| \le M$, $|\Delta_n(x)| \le 2M$, Δ_n is Riemann integrable, and $|f_n(x)| \le M + |c|$ for all n and $x \in (0, 1)$.

c. Show that there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k}(q)$ converges to a value $f(q)$ for all rational $q \in [0, 1]$. (Hint: enumerate the rationals in $[0, 1]$ as $q_1, q_2, q_3, ...$ First find a subsequence $f_{n_k}^{(1)}(q_1)$ which converges, which is possible by the Bolzano-Weistrass theorem. Next find a subsequence $f_{n_k}^{(2)}(q_2)$ which is a sub-subsequence of $f_{n_k}^{(1)}$, and which converges. Iterate this. As the final subsequence, choose $f_{n_k}^{(k)}$ and check that this converges at each rational.)

d. Check that the subsequence $\{f_{n_k}\}$ which you found in part c. converges uniformly to a continuous function $f(x)$ on $[0, 1]$, in the sense that, as $k \to \infty$, $\sup_{x \in [0,1]} |f(x) - f_{n_k}(x)| \to 0$. It will be helpful to use the bound $|f_{n_k}'(x)| \le M$ for all x and all k.

e. Check that ϕ is uniformly continuous on $0 \le x \le 1$, $|y| \le M + |c|$, and hence

$$\phi(t, f_{n_k}(t)) \to \phi(t, f(t))$$

uniformly on $[0, 1]$.

f. Check that $\Delta_n(t)$ converges to 0 uniformly on $[0, 1]$, since

$$\Delta_n(t) = \phi(x_i, f_n(x_i)) - \phi(t, f_n(t))$$

in (x_i, x_{i+1}).

g. Hence

$$f(x) = c + \int_0^x \phi(t, f(t))dt.$$

Thus $f(x)$ is a solution to the given problem.

Solution.

a. Let $x_i = \frac{i}{n}$, $0 \le i \le n$. Define, recursively, for $i = 0, 1, 2, ..., n-1$, for $t \in [x_i, x_{i+1}]$,

$$f_n(t) = c + \frac{1}{n}\sum_{j=0}^{i-1} \phi(x_j, f_n(x_j)) + (t - x_i)\phi(x_i, f_n(x_i)).$$

Thus f_n is piecewise linear, and continuous at each x_i. Furthermore, for $t \in (x_i, x_{i+1})$, $f_n'(t) = \phi(x_i, f_n(x_i))$. Since the derivative is constant on intervals between $[x_i, x_{i+1}]$, it is Riemann integrable, and its indefinite integral is equal to the continuous piecewise linear function f_n up to a constant. It follows that

$$f_n(x) = c + \int_0^x f_n'(t)dt = c + \int_0^x [\phi(t, f_n(t)) + \Delta_n(t)]dt$$

since the functions agree at 0.

b. For $t \in (x_i, x_{i+1})$, $f_n'(t) = \phi(x_i, f_n(x_i))$, and hence $|f_n'(t)| \le M$. Then

$$|\Delta_n(t)| = |f_n'(t) + \phi(t, f_n(t))| \le 2M.$$

Since f_n is continuous and ϕ is continuous $\phi(t, f_n(t))$ is continuous and hence Riemann integrable. Hence $\Delta_n(t)$ is Riemann integrable since it is the difference between $f_n'(t)$ and $\phi(t, f_n(t))$ except at finitely many points. Since $|\phi(t, f_n(t)) + \Delta_n(t)| \le M$, it follows that

$$f_n(x) \le |c| + M$$

by bounding the integrand from part a by M.

c. Let $q_1, q_2, q_3, ...$ be an enumeration of the rationals in $[0, 1]$. Since $|f_n(q_1)| \le M + |c|$ is bounded there is a subsequence $f_{n_i}^{(1)}(q_1)$ which converges by the Bolzano-Weiestrass Theorem. Next, find a further subsequence $f_{n_i}^{(2)}$ of the subsequence $f_{n_i}^{(1)}$ such that $f_{n_i}^{(2)}(q_2)$ converges. Since the subsequence of a convergent subsequence converges to the same limit, $f_{n_i}^{(2)}(q_1)$ also converges. Iterate this construction to find, for $k = 1, 2, ...$, a sequence $f_{n_i}^{(k)}$ which is a subsequence of $f_{n_i}^{(j)}$ for $1 \le j \le k-1$, such that

$f_{n_i}^{(k)}$ converges at $q_1, q_2, ..., q_k$. The sequence f_{n_i} whose ith term is the ith term of $f^{(i)}$ has all terms after the ith belonging to $f^{(i)}$, and hence is a subsequence of the original sequence, which converges at $q_1, q_2, q_3, ...$

d. Given $\epsilon > 0$, let $\delta = \frac{\epsilon}{3M}$ and let $0 = x_0 < x_1 < ... < x_m = 1$ be a δ-fine partition of $[0, 1]$ consisting of rational numbers. Choose N sufficiently large so that $j, k > N$ implies that $|f_{n_j}(x_i) - f_{n_k}(x_i)| < \frac{\epsilon}{3}$ for each $i = 0, 1, 2, ..., m$. Given $t \in [0, 1]$, let $t \in [x_i, x_{i+1}]$, we have, from the integral representation,

$$|f_{n_j}(t) - f_{n_j}(x_i)| = \left| \int_{x_i}^{t} \phi(u, f_{n_j}(u)) + \Delta_{n_j}(u)) du \right| \leq M(t - x_i) \leq M\delta \leq \frac{\epsilon}{3}$$

and similarly for $|f_{n_k}(t) - f_{n_k}(x_i)|$. Hence

$$|f_{n_j}(t) - f_{n_k}(t)| \leq |f_{n_j}(t) - f_{n_j}(x_i)| + |f_{n_k}(t) - f_{n_k}(x_i)| + |f_{n_j}(x_i) - f_{n_k}(x_i)| < \epsilon.$$

It follows that (f_{n_j}) is uniformly Cauchy, and hence converges uniformly to a continuous function f.

e. Since $\{(x, y) : 0 \leq x \leq 1, |y| \leq M + |c|\}$ is closed and bounded and ϕ is continuous on this space, it is uniformly continuous. Thus, given $\epsilon > 0$ there is $\delta > 0$ such that if

$$\|(x, y) - (x_0, y_0)\|_2 < \delta$$

then $|\phi(x, y) - \phi(x_0, y_0)| < \epsilon$. Since f_{n_k} converges to f uniformly, there is a N such that $k > N$ implies for all $t \in [0, 1]$, $|f_{n_k}(t) - f(t)| < \delta$. Then $\|(t, f_{n_k}(t)) - (t, f(t))\|_2 < \delta$, so $|\phi(t, f_{n_k}(t)) - \phi(t, f(t))| < \epsilon$.

f. For fixed n, for $t \in (x_i, x_{i+1})$, $|t - x_i| < \frac{1}{n}$, and hence $|f_n(t) - f_n(x_i)| \leq \frac{M}{n}$. Hence $\|(x_i, f_n(x_i)) - (t, f_n(t))\|_2 \leq (M + 1)\frac{1}{n}$ tends to 0 uniformly in t as $n \to \infty$. Thus, by the uniform continuity of ϕ, $\Delta_n(t) = \phi(x_i, f_n(x_i)) - \phi(t, f_n(t))$ tends to 0 uniformly as $n \to \infty$.

g. Since $\Delta_n(t) \to 0$ uniformly in t as $n \to \infty$, and $\phi(t, f_{n_k}(t)) \to \phi(t, f(t))$ uniformly in t as $k \to \infty$,

$$\begin{aligned} f(x) &= \lim_{k \to \infty} f_{n_k}(x) \\ &= \lim_{k \to \infty} c + \int_0^x \phi(t, f_{n_k}(t)) + \Delta_{n_k}(t) dt \\ &= c + \int_0^x \phi(t, f(t)) dt. \end{aligned}$$

Thus f solves the initial value problem, as may be verified from the Fundamental Theorem of Calculus.