Spring 2017 MAT 536, Complex Analysis

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Homework #11, due in class Wed April 26

Recall that $\Omega \subsetneq \mathbb{C}$ is open connected, $\mathcal{O}(\Omega)$ denotes the set of holomorphic functions on Ω, $\mathcal{H}(\Omega)$ denotes the set of harmonic functions on Ω, and $\mathcal{SH}(\Omega)$ denotes the set of subharmonic functions on Ω; $\Delta_r(z)$ denotes the open disk around z of radius r.

Problem 1. Prove that any $u \in \mathcal{SH}(\Delta_{1+\epsilon}(0))$ is integrable on Δ (start by proving that upper semicontinuous functions not taking value $-\infty$ are integrable, then supply the details of the proof given in class).

Problem 2. Suppose $\mu \in \mathcal{C}^0(\partial\Delta)$ is a continuous real function on the unit circle. Let then

$$\phi(z) := \int_{\partial\Delta} \mu(w) \ln|z-w||dw|$$

be the so-called logarithmic potential. Prove that $\phi \in \mathcal{H}(\mathbb{C} \setminus \overline{\Delta})$, and prove that if $\mu \equiv 1$, then $\phi|_\Delta$ is constant.

Problem 3. Suppose that both $f(z)$ and $zf(z)$ are complex-valued harmonic functions (i.e. $\Delta f \equiv \Delta(zf) \equiv 0$). Prove that f is holomorphic.

Problem 4. Suppose $u_n \in \mathcal{SH}(\Delta)$ is a sequence of functions such that $\sup_{z\in K, n\in\mathbb{N}} u_n(z) < +\infty$ for any compact $K \subset \Omega$. Prove that $\sum 2^{-n}u_n$ is subharmonic. *(Hint: upper semi-continuity follows from Fatou's lemma)*

Problem 5. Let $\{z_n\} \subset \Delta$ be a countable dense set. Show that

$$u(z) := \sum 2^{-n} \ln|z - z_n|$$

is a subharmonic function on Δ that is discontinuous almost everywhere on Δ.