Spring 2017 MAT 536, Complex Analysis
Instructor: Samuel Grushevsky
Homework #4, due in class Wed February 22

Problem 1. Suppose $f, g : \mathbb{C} \to \mathbb{C}$ are holomorphic functions. Prove that if for any $z \in \mathbb{C}$ the inequality $|f(z)| \le |g(z)|$ holds, then there exists a constant $c \in \mathbb{C}$ such that $f(z) = cg(z)$ for any $z \in \mathbb{C}$.

Notation. Let $\Delta = \{|\zeta| < 1\} \subset \mathbb{C}$ be the open unit disk, let $\gamma = \{|\zeta| = 1\} = \partial\Delta$ be the unit circle, and let $\Delta' := \mathbb{C}\backslash(\Delta \sqcup \gamma) = \{|\zeta| > 1\}$ be the complement of the closed unit disk.

Problem 2. Suppose $f : \Delta \to \mathbb{C}$ is a holomorphic function which has a pole at $z = 1$. Prove that the power series expansion for f centered at 0 does not converge at any point $z \in \gamma$.

Problem 3. Let $f : \gamma \to \mathbb{C}$ be a *continuous* function. For $z \in \mathbb{C} \setminus \gamma$ define the function $F(z)$ by the Cauchy integral formula, that is let

$$F(z) := \frac{1}{2\pi i} \int_{\zeta \in \gamma} \frac{f(\zeta) d\zeta}{\zeta - z}.$$

Prove that $F(z)$ is a holomorphic function on $\mathbb{C} \setminus \gamma$, and prove that $\lim_{z \to \infty} F(z) = 0$.

Problem 4 (20 points)**.** Continuing in the setup of the previous problem, and assuming that $f \in C^1(\gamma)$ (you may assume smooth if you wish),

(a) Prove that for any $\zeta_0 \in \gamma$ there exist limits

$$F^-(\zeta_0) := \lim_{z \in \Delta, z \to \zeta_0} F(z) \qquad \text{and} \qquad F^+(\zeta_0) := \lim_{z \in \Delta', z \to \zeta_0} F(z)$$

(b) Prove the *Sokhotski-Plemelj formula*, i.e. prove that the equality

$$F^+(\zeta_0) - F^-(\zeta_0) = \pm f(\zeta_0)$$

holds for any $\zeta_0 \in \gamma$ (the sign depends on your convention of how γ is oriented).

Hint: Express F as

$$F(z) = \frac{1}{2\pi i} \int_{\zeta \in \gamma} \frac{f(\zeta) - f(\zeta_0)}{\zeta - z} d\zeta \ + \ \frac{f(\zeta_0)}{2\pi i} \int_{\zeta \in \gamma} \frac{d\zeta}{\zeta - z}$$