MAT 118 Spring 2017

Practice Exam for Midterm #1

With solutions! (Last two problems omitted.)

1. Consider this preference schedule for an election between candidates A, B and C:

# voters:	5	4	3
1st	A	C	B
2nd	B	B	C
3rd	C	A	A

(a) Which candidate wins using the plurality method?
A wins, since A has the most 1st votes.

(b) Use the weighted Borda count method to determine the outcome, in which 1st place votes are each 4 points, 2nd place votes are 2 points, and 3rd place votes are 1 point.
A got 5 1st places votes, 0 2nd place votes and 7 3rd place votes. So:
A gets $5 \times 4 + 0 \times 2 + 7 \times 1 = 27$ points. Similarly, we have:
B gets $3 \times 4 + 9 \times 2 + 0 \times 1 = 30$ points.
C gets $4 \times 4 + 3 \times 2 + 5 \times 1 = 27$ points.
So B is ranked 1st, and A and C are tied for 2nd.

(c) Use the plurality-with-elimination method to determine the outcome.
The first round involves comparing the candidates using 1st place votes.
A has 5, B has 3, and C has 4. No one has a majority, so we eliminate the candidate with the fewest amount of 1st place votes: in this case it is B. But the 3 votes that were 1st place votes for B had C ranked as 2nd, so these 3 votes are transferred to C. The next round: A still has 5 and C now has $4 + 3 = 7$. Now C has a majority, so wins the election.

(d) Finally, apply the method of pairwise comparisons to determine the outcome.
A v B: 5 voters prefer A over B, and 7 voters prefer B over A, so B gets 1 point.
A v C: 5 voters prefer A over C, and 7 voters prefer C over A, so C gets 1 point.
B v C: 8 voters prefer B over C, and 4 voters prefer C over B, so B gets 1 point.
1st is B with 2 points, 2nd is C with 1 point, and 3rd is A with 0 points.

(e) Is there a Condorcet candidate? If so, who is it?
Yes, B is a Condorcet candidate.

2. Consider the weighted voting system [20; 7, 6, 4, 3, 1] with players P_1, P_2, P_3, P_4, P_5.

(a) Does any player have veto power? If so, how many and which ones?

Players P_1, P_2, P_3, P_4 have veto power. Recall that a player with weight w has veto power If $V - w < q$, where V = sum of all weights = 7 + 6 + 4 + 3 + 1 = 21, and q = 20 is the quota. We can directly check that 21 - 7 = 14 < 20, 21 - 6 = 15 < 20, 21 - 4 = 17 < 20, and 21 - 3 = 18 < 20, which shows that P_1, P_2, P_3, P_4 have veto power. Since 21 - 1 = 20 is not less than 20, player P_5 does not have veto power.

(b) Compute the Banzhaf power distribution.

The winning coalitions are $\{ \underline{P_1}, \underline{P_2}, \underline{P_3}, \underline{P_4} \}$ and $\{\underline{P_1}, \underline{P_2}, \underline{P_3}, \underline{P_4}\ P_5\}$. The critical players are underlined. The critical counts are thus $B_1 = B_2 = B_3 = B_4 = 2$ and $B_5 = 0$. The total critical count is T = 2 + 2 + 2 + 2 + 0 = 8. Then $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 2/8 = 1/4$ and $\beta_5 = 0/8 = 0$.

(c) How many sequential coalitions are there in this weighted voting system?

In general, in a weighted voting system with N players, there are N! sequential coalitions, where $N! = N \times (N-1) \times (N-2) \times \ldots \times 3 \times 2 \times 1$. Here N = 5, and $5! = 5 \times 4 \times 3 \times 2 \times 1$, which is equal to $20 \times 6 = 120$.

3. Compute the Shapley-Shubik power distribution of [5; 4, 2, 1].

Here are the sequential coalitions, with pivotal players underlined:

$\langle P_1, \underline{P_2}, P_3 \rangle$

$\langle P_1, \underline{P_3}, P_2 \rangle$

$\langle P_2, \underline{P_1}, P_3 \rangle$

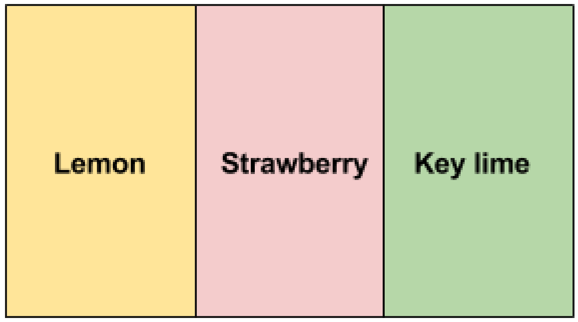
$\langle P_2, P_3, \underline{P_1} \rangle$

$\langle P_3, \underline{P_1}, P_2 \rangle$

$\langle P_3, P_2, \underline{P_1} \rangle$

Thus the pivotal counts are $SS_1 = 4$ and $SS_2 = SS_3 = 1$. The Shapley-Shubik indices are given by these numbers divided by N! = 3! = 6, and we get $\sigma_1 = 4/6 = 2/3$ and $\sigma_2 = \sigma_3 = 1/6$.

4. Andy and Brianna are dividing a giant rectangular ice cream cake using the divider-chooser method. After flipping a coin, they decide that Andy will be the divider. The cake has 3 flavors: lemon, strawberry and key lime. Here's an illustration of the cake:



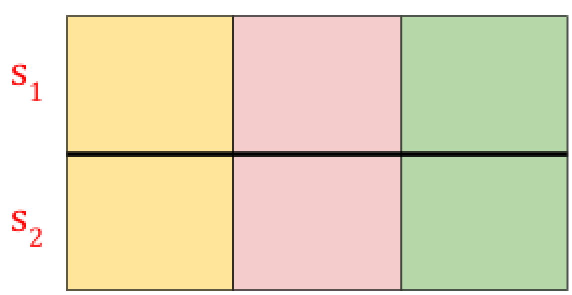
Andy likes does not like strawberry at all and likes lemon twice as much as key lime, while Brianna likes all flavors equally.

(a) Make a table listing the values, relative to the total value of the cake, of each flavor according to each of Andy and Brianna. Use either fractions or percentages.

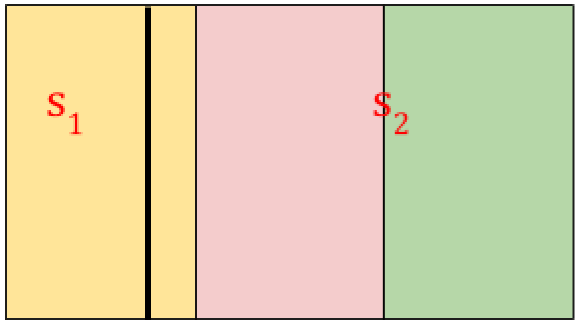
	L	S	KL
A	2/3	0	1/3
B	1/3	1/3	1/3

(b) Describe, with pictures and in words (i.e. fractions of flavors), **two** distinct ways that Andy can divide the cake, with the following restriction: at least one of the divisions must allow Brianna to leave with *more* than a minimal fair share, in her opinion.

First cut: A can cut horizontally, so that both s_1 and s_2 each contain half of every flavor:



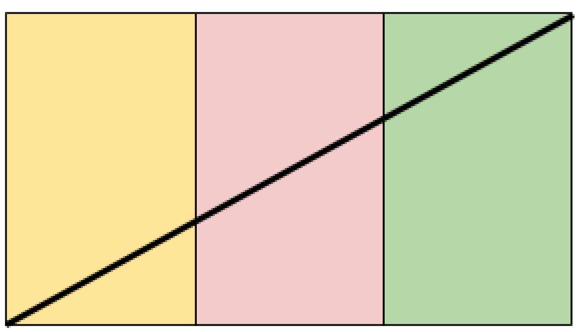
Second cut: A can cut vertically within the lemon so that s_1 consists of 3/4 of the lemon portion and s_2 contains 1/4 of the lemon portion plus the rest of the cake:



This works since A values s_1 at 3/4 times his value of lemon, which is 2/3. Thus his value of s_1 is equal to (3/4) × (2/3) = 1/2. If s_1 is worth 1/2, then the remaining part of s_2 must also be worth 1/2.

Discussion: In case you are wondering how to get the exact cut above, here is a general method. We know that since A values the lemon portion at 2/3 the value of the total cake, which is more than 1/2, there is some fraction, say N, of lemon, that is worth 1/2 to A. We know from above that N = 3/4 works, but how do we come up with this value of N? We can do the following. The only constraint on N is that the cut must be such that A values the resulting piece of lemon at 1/2. This leads to the equation N × (2/3) = 1/2. Indeed, the left hand side of this equation is how much lemon is in the piece (which is N) multiplied by A's value of the lemon. Solving this equation yields N = 3/4.

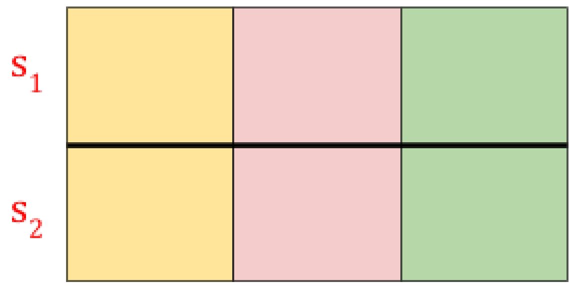
Warning: The following cut is invalid:



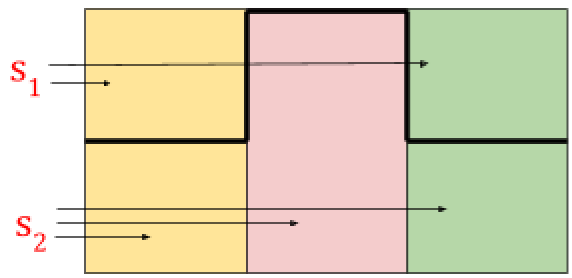
This is because A does not see the two resulting pieces as equal. The top part with more lemon will be more valuable to A.

Remark: The following answer to question 4(b) is also valid:

First cut: Same as the first cut above:



Second cut: Similar to the first cut, but change how the strawberry is divided:



In words, s_1 is 1/2 of the lemon portion and 1/2 of the key lime portion, but no strawberry, and s_2 is 1/2 of the lemon, 1/2 of the key lime, and all of the strawberry.

5. In problem #5, suppose that we have a third player, Casey, in addition to Andy and Brianna. Casey only likes lemon. Now suppose the 3 players decide to use the Lone-divider method.

(a) Write a table, enlarging the table from 6(a), listing the values, relative to the total value of the cake, of each flavor according to the 3 players. Use either fractions or percentages.

	L	S	KL
A	2/3	0	1/3
B	1/3	1/3	1/3
C	1	0	0

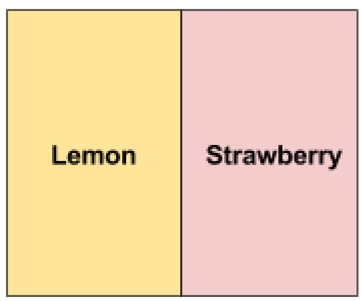
(b) Suppose Brianna is the divider, and that she divides the cake along the flavor lines, so that one piece is all lemon, one is all strawberry, and one is all key lime. What are the bidding lists (i.e. fair share lists) of Andy and Casey?

Say the shares are labelled such that s_1 is all the lemon, s_2 is all the strawberry, and s_3 is all the key lime. Then the value table for the players regarding these shares is the same as the table in part (a), replacing L, S, KL by s_1 , s_2, s_3 respectively. A fair share in this context is a share worth at least 1/3 value to a player. Thus the bidding list for A is { s_1 , s_3} and the bidding list for C is { s_1 }.

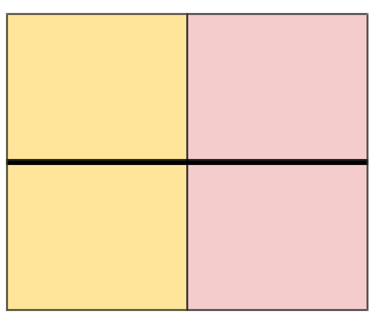
(c) Describe how this instance of the Lone-divider method might be completed.

We can give s_1 to C, s_2 to B, and s_3 to A.

Discussion: On the actual exam, be prepared for the possibility that the pieces at this last stage cannot be fairly divided, and that we might have to recombine assets. For example: <u>if A had the same preferences as C</u>, and only liked lemon, then the bidding lists would instead be { s_1 } for *both* A and C. Then we cannot fairly distribute the shares, since both A and C want only s_1. So the next step is to give B any share that is not s_1 , say s_3 . Then B leaves, and A and C recombine s_1 and s_2 and use the divider-chooser method on this recombined asset, depicted below:



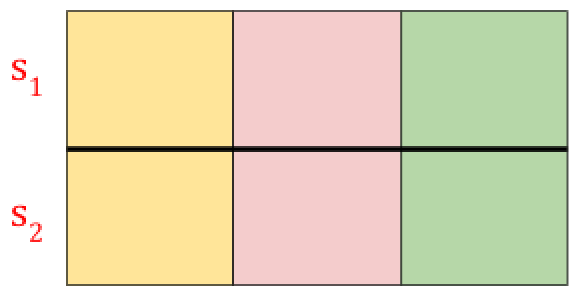
Now say A is the new divider. A valid cut for A to make is a horizontal cut:



Then C, seeing both parts as the same, takes either one, say the top. In summary, A and C each get 1/2 the lemon portion and 1/2 the strawberry, while B gets all the key lime. Note that A and C leave with what they value as 1/2 the value of the cake in this scenario!

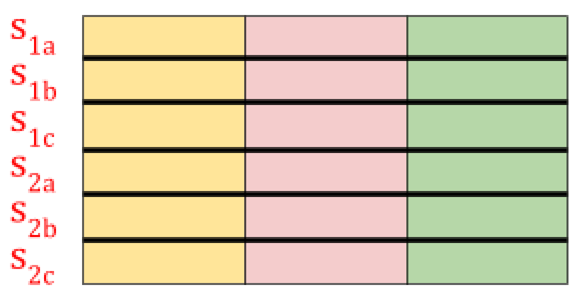
6. Describe how the Lone-Chooser method might proceed with the cake in #6, in which Casey is the chooser. In your description, how much value does she leave with?

Step 1: A and B use the divider-chooser method to divide the cake into two parts. For this we can suppose that A is the divider (you can choose either one) and we can use any of the cuts we gave in problem 4(b). We suppose A makes the cut



B, not caring between the two pieces, takes the bottom part.

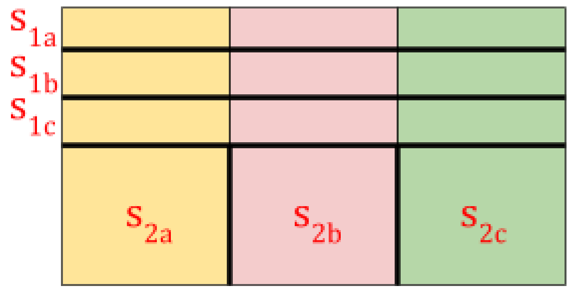
Step 2: Each of A and B have to divide their portion into what they consider three equal shares. The easiest way this might proceed is as follows:



In other words, A has shares s_{1a} , s_{1b} , s_{1c} and B has shares s_{2a} , s_{2b} , s_{2c} . Each of these shares is indistinguishable to all players! So C randomly chooses s_{1a} and s_{2a} . In the end, A has s_{1b} , s_{1c} , while B has s_{2b} , s_{2c} , and C has s_{1a} , s_{2a} . In this scenario, all three players end up with what they see as 1/3 the total value of the cake.

Remark: The following answer to question 6 is also valid:

Since B does not distinguish between flavors, she can divide her bottom share differently:



As before, C randomly chooses s_{1a}. But when it comes to choosing from s_{2a} , s_{2b}, s_{2c}, the choice is not random: C clearly prefers s_{2a} , and will take that piece. In the end, A has s_{1b}, s_{1c} , while B has s_{2b}, s_{2c} , and C has s_{1a}, s_{2a}. But in this scenario, while A and B end up with what they see as 1/3 of the total value, C ends up with 2/3 of value!

To see this last point, first recall that C only cares about how much lemon she gets. Note that s_{1a} has 1/3 of half of the original lemon portion, so has 1/6 of the total lemon portion. On the other hand, s_{2a} has 1/2 of the total lemon portion. So 1/2 + 1/6 = 4/6 = 2/3.