Ch. 1 1.5, 1.6 Method of Pairwise Comparisons, Arrows Impossibility Theorem

The method of Pairwise Comparisons

- every player (candidate) plays every other player once
- the winner of each match gets 1 point, the loser gets 0 pts, and if there is a tie, each get ½ pt.
- the player with the most points wins.

Using our preference schedule from earlier lectures:

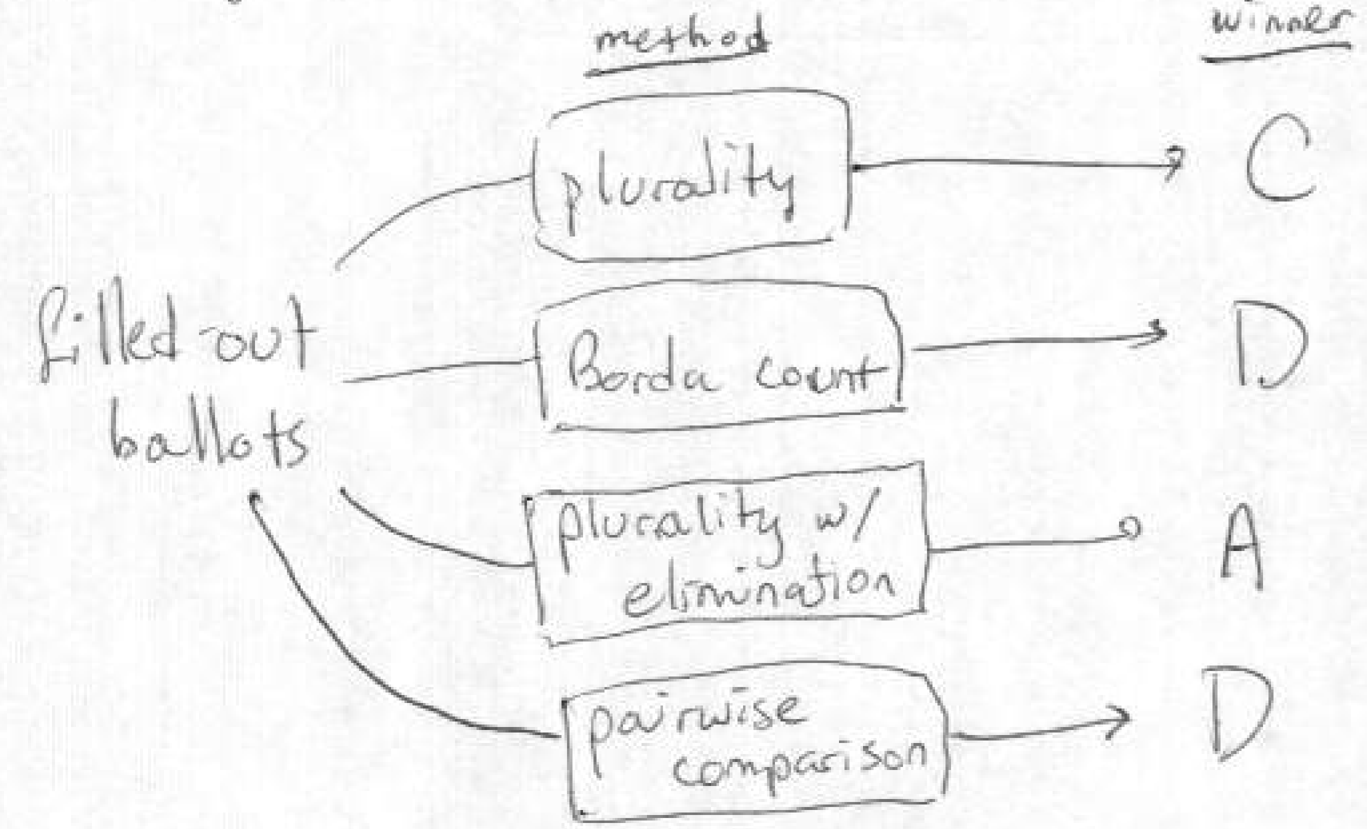
pair	# voters	winner
A v B	10 v 20	B
A v C	19 v 11	A
A v D	10 v 20	D
B v C	11 v 19	C
B v D	8 v 22	D
C v D	13 v 17	D

Total # points:

A	B	C	D
1	1	1	3

Thus D wins.

A summary of our 4 methods with this example:



The method of pairwise comparisons can take a very long time to compute if the number of candidates, N, is large. (A computer can easily do it for large N, though.)

$$\#\text{ pairings among } N \text{ candidates} = \frac{N(N-1)}{2}.$$

Arrow's Impossibility Theorem

("No voting method is perfect")

How do we measure the quality of a voting method?

One way is to consider a set of "fairness criteria" and see if a voting method satisfies these criteria.

Some fairness criteria:

- the majority criterion:

  A majority candidate should always win.

- the Condorcet criterion:

  A candidate is a "condorcet candidate" if it wins in any pairwise comparison.
  This criterion says a condorcet candidate should always win.

- the monotonicity criterion:

  if a candidate wins, they should still win if a voter highers their preference for that candidate.

(4)

- the <u>independence of irrelevant alternatives (IIA)</u> <u>criterion</u>:

  if a candidate wins, they should still win if a losing candidate (or a group of losing candidates) is removed from the election.

Arrow's impossibility theorem says that there is no voting method that satisfies all the fairness criteria.

We will not prove this theorem, but only show how <u>none</u> of the 4 voting methods we've learned about satisfy all these criteria.

In our election from lecture 1, recall that C won with the plurality method, but that if A and B were omitted from the election, then D won instead. Thus the plurality method violates the IIA criterion.

↓ Criterion	Plurality	Borda Count	Plurality w/Elimination	Pairwise Comparisons
Majority				
Condorcet				
Monotonicity				
IIA	X			

The "X" means the voting method violates the criterion, while a "✓" will mean that the method always satisfies the criterion.

If a candidate has a majority of votes, the plurality method will choose him/her as the winner. Thus plurality always satisfies the majority criterion.

(6) The same goes for plurality with elimination and also pairwise comparisons.

Criterion	Plurality	Borda Count	Plurality w/ Elimination	Pairwise Comparisons
majority	✓		✓	✓
Condorcet				
monotonicity				
IIA	X			

The Borda Count method violates the majority criterion:

#votes →	6	2	3
1st	A	B	C
2nd	B	C	D
3rd	C	D	B
4th	D	A	A

#points for A: $4 \times 6 + 1 \times 5 = 29$

— B: $4 \times 2 + 3 \times 6 + 2 \times 3 = 8 + 18 + 6 = 32$

— C: $4 \times 3 + 3 \times 2 + 2 \times 6 = 12 + 6 + 12 = 30$

— D: $3 \times 3 + 2 \times 2 + 1 \times 6 = 9 + 4 + 6 = 19$

A has a majority of 6 out of 11 votes, but B wins with the Borda Count method.

The plurality method violates the Condorcet condition:

	6	5	2
1st	A	B	C
2nd	B	C	B
3rd	C	A	A

Plurality chooses A. However:

A v B $\longrightarrow$ 6 v 7 $\longrightarrow$ B wins over A

B v C $\longrightarrow$ 11 v 2 $\longrightarrow$ B wins over C

So that B is a Condorcet candidate.

Plurality-with-Elimination violates the monotonicity criterion:

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	7	8	10	2
1^{st}	A	B	C	A
2^{nd}	B	C	A	C
3^{rd}	C	A	B	B

round 1:

A	B	C
9	8	10

round 2:

A	~~B~~	C
9		18

$\longrightarrow$ C wins.

Now, suppose the two voters that voted

1	A
2	C
3	B

put C ahead of A, i.e.

1	C
2	A
3	B

.

Then we have

	7	8	12
1^{st}	A	B	C
2^{nd}	B	C	A
3^{rd}	C	A	B

round 1:

A	B	C
7	8	12

round 2:

~~A~~	B	C
	15	12

$\downarrow$

B wins, not C!

Pairwise comparisons violates the IIA: 9

	2	5	3
1st	A	B	C
2nd	B	C	A
3rd	C	A	B

$A \vee B \to 5 \vee 5 \to$ tie, each of A and B get $\frac{1}{2}$ point

$B \vee C \to 7 \vee 3 \to$ B gets 1 point

$A \vee C \to 2 \vee 8 \to$ C gets 1 point

B wins with $1 + \frac{1}{2} = \frac{3}{2}$ ~~votes~~ points, against A with 0 ~~votes~~ points and C with 1 point

If we remove C from the election, then instead A and B tie.

(10) We can continue to fill out the whole table

Criterion	Plurality	Borda Count	Plurality w/Elimination	Pairwise Comparisons
majority	✓	X	✓	✓
condorcet	X	X	X	✓
monotonicity	✓	✓	X	✓
IIA	X	X	X	X