Ch 2: The Mathematics of power

Last chapter we learned about elections. One obvious example was not mentioned: the U.S. presidential electio What kind of voting method is used?

We all know that because of the electoral college, the U.S. presidential election does not use the plurality method. We can go through the other 3 voting methods we learned to see that it uses none of them.

Making an ideal assumption, which usually holds in reality, that all electoral college votes for a state are for whichever candidate got the most votes in that state, we can view the U.S. presidential election as a two-step process:

- each state holds a plurality method election to determine the winner of that state
- each state then "votes", voting for the candidate that won that state

2 The second step needs more explaining.
In actuality, each state puts forth a certain amount of votes, i.e. NY has 29

TX has 38

KY has 8

CA has 55

An election in which each voter (here each "state") has a certain number of votes to use, or a number of points, involves what we call "weighted voting".

A simpler example of weighted voting:

There is a mythical society in which age is revered above all else. Here, whenever anyone votes in an election, they can cast as many votes as their age in years.

In any voting system we will refer to the voters as <u>players</u>, often.

We'll write N = #players in a given weighted voting system,

and $P_1, P_2, \ldots, P_N$ for the names of the players.

We write $w_1, w_2, \ldots, w_N$ for the weights of $P_1, P_2, \ldots, P_N$, respectively.

The total # votes is $V = w_1 + w_2 + \cdots + w_N$.

q = Quota = minimum # votes to pass a motion.

A generic weighted voting system as above will be denoted

$$[q : w_1, w_2, \ldots, w_N]$$

where $w_1 \geq w_2 \geq w_3 \geq \ldots \geq w_N$.

We can try to write the example with the U.S. presidential election in this format. However, there are 50 players, the states, which is a bit muc

(4) Nonetheless we can say a little about what it would look like.

The condition $w_1 \geq w_2 \geq \cdots \geq w_N$ means P_1 should have the most electoral college votes, then P_2, and so forth. So,

$$P_1 = CA, \quad w_1 = 55$$

$$P_2 = TX, \quad w_2 = 38$$

$$P_3 = FL, \quad w_3 = 29$$

$$P_4 = NY, \quad w_4 = 29$$

$$P_5 = IL, \quad w_5 = 20$$

$$P_6 = PA, \quad w_6 = 20$$

etc.

$$V = w_1 + w_2 + \cdots + w_{50} = \text{total \# electoral college votes} = 538$$

$$q = \frac{538}{2} + 1 = 270 \quad \left(\text{the quota } q \text{ is a simple majority}\right)$$

Thus this weighted voting system in the above introduced notation looks like

$$[270 : 55, 38, 29, 29, 20, 20, \ldots]$$

The point of this notation is that it contains all the data we need for our analysis of the system.

For our weighted voting systems, the players will be voting on a "motion", and each player votes either "yes" or "no" on whether to pass the motion.

As another example, consider our distopyian society in which each citizen gets as many votes as their age in years. Suppose in a town hall 3 people, aged 16, 20, 30 are voting on a motion. Suppose further that the quota $q = 30$. This system may be written

$$[30 : 30, 20, 16]$$

(6) There's a problem here: P_1 can vote No
P_2, P_3 can vote YES

(or vice versa), and both NO and YES get enough votes to pass the quota!

When in a voting system it is possible to have both No and YES reach the quota, we say that the system has <u>anarchy</u>.

One way to guarantee this does not happen is to require that $q > V/2$, i.e. the quota is at least a majority of the total # of votes.

Suppose in the above example that we instead have $q = 70$, so that we are considering the system

$$[70: 30, 20, 16]$$

Then no motion can pass! This is a <u>gridlock</u>.
To avoid this, we need $q \leq V$,

i.e. the quota should be no more than the total # of votes in the system.

Example: Consider the weighted voting system

$$[19 : 8, 7, 3, 2].$$

Note that $V = 8+7+3+2 = 20$.

Even though $q < V$, all four players need to vote for a motion to pass it.

We see that this weighted voting system is effectively the same as the system

$$[4 : 1, 1, 1, 1].$$

Example: Consider $[11 : 12, 5, 4]$.

Here a motion can only pass if and only if P_1 votes for it! In this situation, P_1 is a dictator.

Example: Consider $[12 : 9, 5, 4, 2]$.

In this system, a motion can only pass if P_1 votes for it. We say that P_1 has veto power.

Unlike a dictator, a player with veto power can not necessarily pass a motion by him/herself.

More formally:

- A player is a dictator if their weight is bigger than or equal to the quota.

- A player has veto power if his/her weight w satisfies
$$w < q \quad \text{and} \quad V - w < q.$$

(Thus we use the convention that a dictator does not "have veto power".)