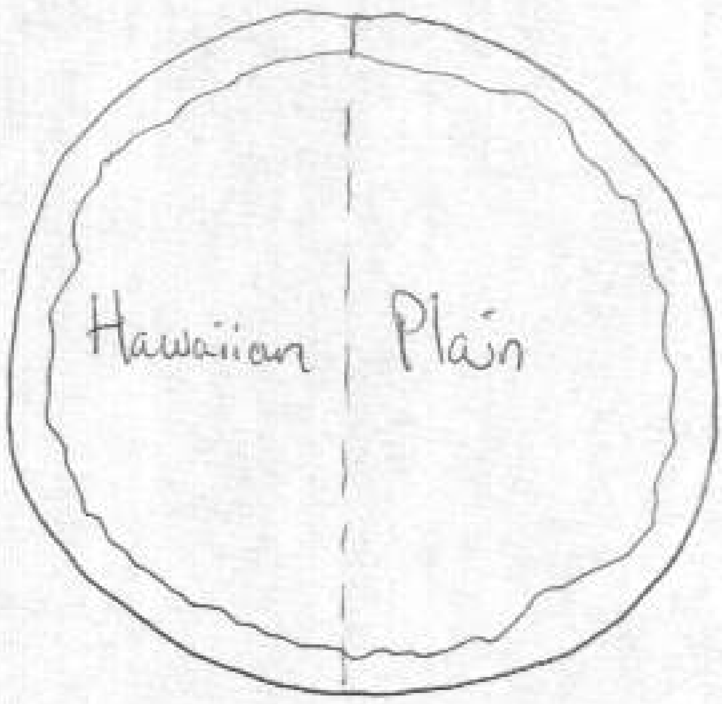
Chapter 3: The Mathematics of Sharing

Two very hungry people, Dan and Clare, arrive at a pizzeria just before closing. There is only one pizza left for sale:



A half plain - half Hawaiian pie. They need a way to divide the pizza fairly.

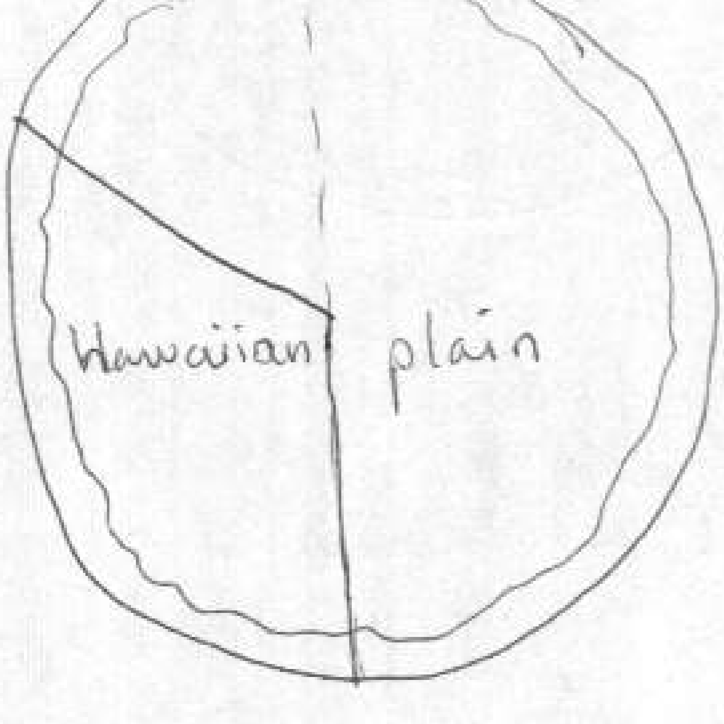
The owner sees the natural division in the pie and suggests that one of them takes the Hawaiian half and the other one takes the plain half. Dan objects — what if both he and Clare prefer Hawaiian? Then one person leaves with a half they did not want.

(2)

Having taken MAT118 in the past, Dan knows a good method for dividing the pie.

First, Dan cuts the pizza into two halves that he considers, by his preferences, to be equal.

For example if he prefers Hawaiian "3 times as much" at plain, then the following, in his eyes, is an equal division:



i.e. $\frac{2}{3}$ of the Hawaiian and $\frac{1}{3}$ of the Hawaiian + <u>all</u> the plain.

This is because in his eyes, the Hawaiian portion is really $\frac{3}{4}$ worth of the pie (i.e. 75%), so that

$\frac{2}{3} \times \frac{3}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{2}{3}$ of the Hawaiian is worth $\frac{1}{2}$ to him

The next step, Dan says, is to let Clare choose the divided portion of her choice.

Clare actually values the Hawaiian and plain portions equally, so she obviously chooses the part which has $\frac{1}{3}$ of the Hawaiian and the plain half. Clare really lucked out — she is walking away with what she sees as more than $\frac{1}{2}$ the pizza! (To her it is $\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{2}=\frac{4}{6}=\frac{2}{3}$

In this example, Dan was the *divider* and Clare the *chooser*, and they used what we call the *divider-chooser method*.

The above is an example of a *fair division game*. The elements of such a game are:

- the *assets*: whatever is being divided. In the above example, the pizza. In real life applications, they are real estate, art, jewelry, money, or less tangible things like permissions or rights.

- the players: the parties that have a right to some portion of the assets.
  Above: Dan and Clare.
  (Could also be institutions, not people.)

- the value systems: the players each have the ability to assign values to the assets.
  Above: Dan valued Hawaiian pizza 3 times as much as plain, while Clare valued them equally.

A fair division method is a way of dividing the assets among the players in a "fair" way.

More precisely:

Suppose we are given a set of players $P_1, P_2, \ldots, P_N$ and a set of assets S.

A fair share to a player, or proportional fair share, is a share s of the set of assets S such that in the player's opinion, s has value at least $\frac{1}{N}$th of the total value of S.

A fair division of the set of assets S is a division of S into N shares $s_1, s_2, \ldots, s_N$ such that s_i is a fair share to P_i $(i=1,\ldots,n)$.

So a fair-division method is a set of rules that guarantees a fair division of the assets.

The divider-chooser method is a fair-division method for two players dividing a continuous asset.

A continuous asset is something that can be divided in any way, i.e. there are no restrictions on the percentages of division.

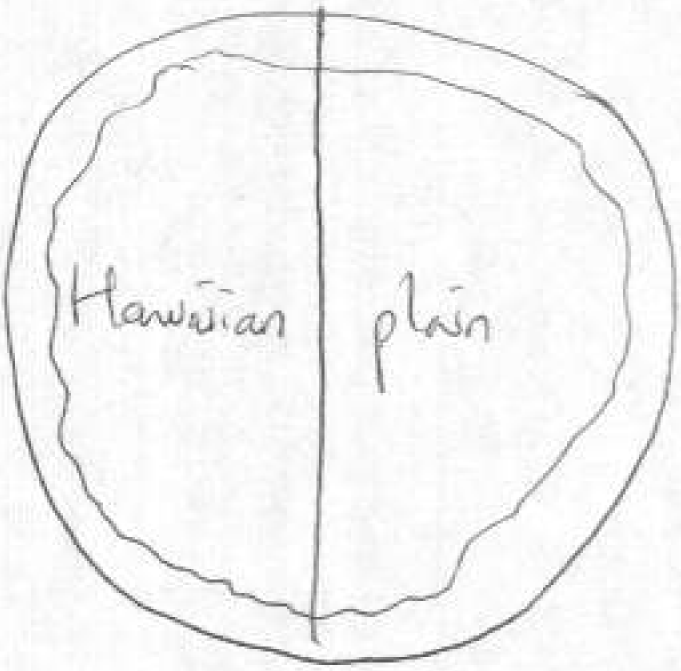
(6) The pizza is considered a continuous asset, and so is land, for example.

An asset or set of assets is <u>discrete</u> if it cannot be divided into arbitrarily small portions.

Examples: paintings, houses, jewelry.

Another example of the divider-chooser method:

Now Clare is the divider, and does:



i.e, she cuts the Hawaiian and plain into separate portions (just as the owner suggested).

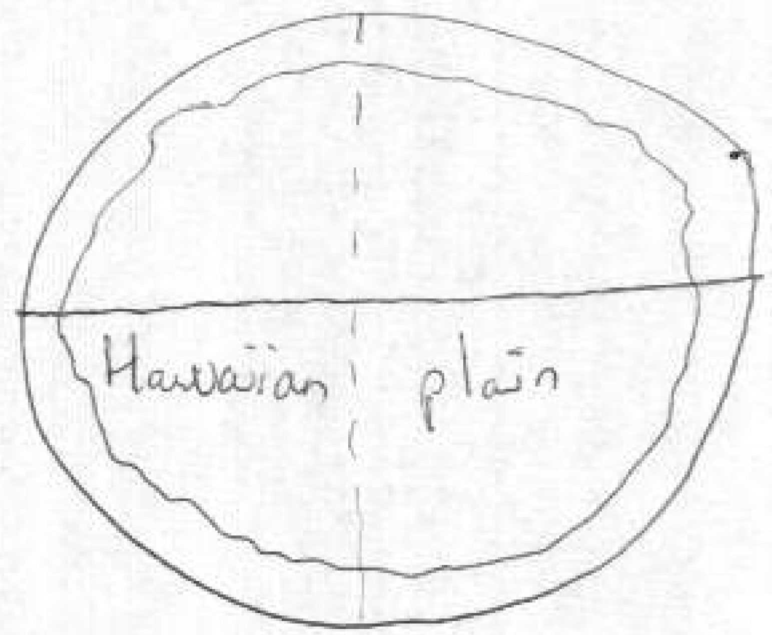
To her eyes, this is an equal division, since she has no preference between Hawaiian & plain.

Now Dan chooses, of course, the Hawaiian portion.

So Clare walks away with what she sees as half, while Dan gets $3/4$!

We see that with the divider-chooser method, the divider always walks away with $1/2$ of what the total value is, in their opinion, while the chooser gets <u>at least</u> half, but often more.

In our example, there is a more naïve division method: cut the pie horizontally so that each player (Dan & Clare) get $\frac{1}{2}$ the Hawaiian part and $1/2$ the plain part:



Regardless of how Dan and Clare value Hawaiian vs. plain, they each get what they see as half the total value.

(8)

This method is not as good as the divider-chooser method, which often leaves one of the players with more than what they considered fair. (And the other player gets $\frac{1}{2}$.)