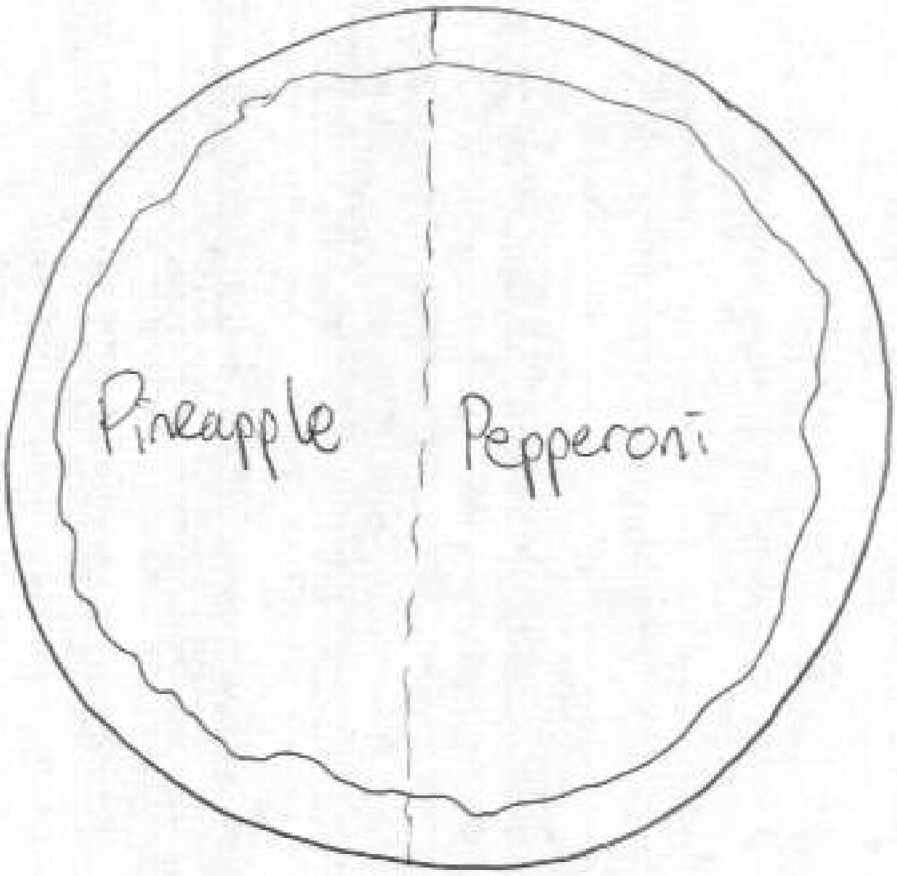
Ch.3 The Lone-Divider Method



Dan, Clare & Charles walk into a pizzeria about to close, and there is only one pizza left (above). They will use the Lone-divider method to fairly divide the pizza.

Here are their preferences:

	Pineapple	Pepperoni
Dan	3/4	1/4
Clare	1/2	1/2
Charles	0	1

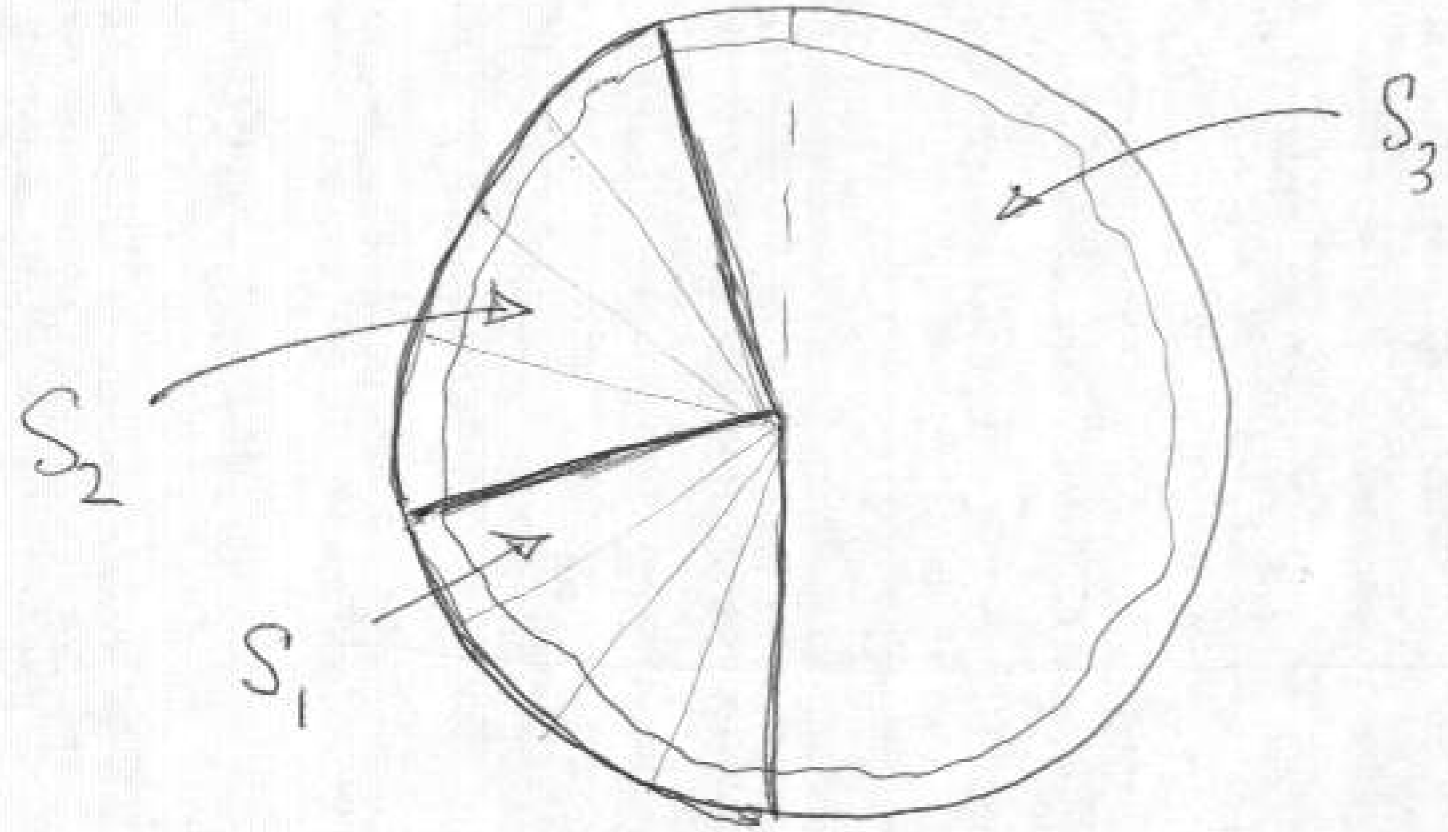
(2)

In words, Dan sees the pineapple part of the pizza as worth 3/4 of the total pie, and so on.

Dan is determined (by drawing straws, for example) to be the <u>divider</u>.

The first step is for him to divide the pizza into 3 parts that for him are of equal value, i.e. each part is $\frac{1}{3}$ of the total worth.

Here is what he does:



i.e. S_1 and S_2 are each $\frac{4}{9}$ of the ~~Hawaiian~~ Pineapple half, and S_3 is $\frac{1}{9}$ of the pineapple + all the pepperoni

Each of S_1, S_2, S_3 is worth $\frac{1}{3}$ to Dan:

$$\begin{array}{l}\text{value} \\ \text{of } S_1 \text{ to Dan}\end{array} = \frac{4}{9} \times \left(\begin{array}{l}\text{value of} \\ \text{pineapple to Dan}\end{array} \right)$$

$$= \frac{4}{9} \times \frac{3}{4} = \frac{3}{9} = \frac{1}{3},$$

and similarly for S_2 and S_3.

Now we can determine how much Clare and Charles, the <u>choosers</u>, value S_1, S_2 and S_3:

	S_1	S_2	S_3
Dan	$1/3$	$1/3$	$1/3$
Clare	$2/9$	$2/9$	$5/9$
Charles	0	0	1

For example, the value of S_1 to Clare is computed as follows:

(4) $\text{Value of } S_1 \text{ to Clare} = \frac{4}{9} \times \left(\text{value of pineapple to Clare}\right) = \frac{4}{9} \times \frac{1}{2} = \frac{2}{9}$

and similarly for S_2. Then we also have

$$\text{value of } S_3 \text{ to Clare} = \frac{1}{9} \times \left(\text{value of pineapple to Clare}\right) + \left(\text{value of pepperoni to Clare}\right)$$

$$= \frac{1}{9} \times \frac{1}{2} + \frac{1}{2}$$

$$= \frac{1}{18} + \frac{9}{18} = \frac{10}{18} = 5/9.$$

Alternatively, already knowing her values for S_1 and S_2 determines that of S_3, which is whatever fraction is left over from 1, i.e.

$1 - 2/9 - 2/9 = 5/9.$

The values of S_1 and S_2 to Charles are zero, since he doesn't value pineapple at all. He values S_3 to be the whole pie, since it contains all the pepperoni.

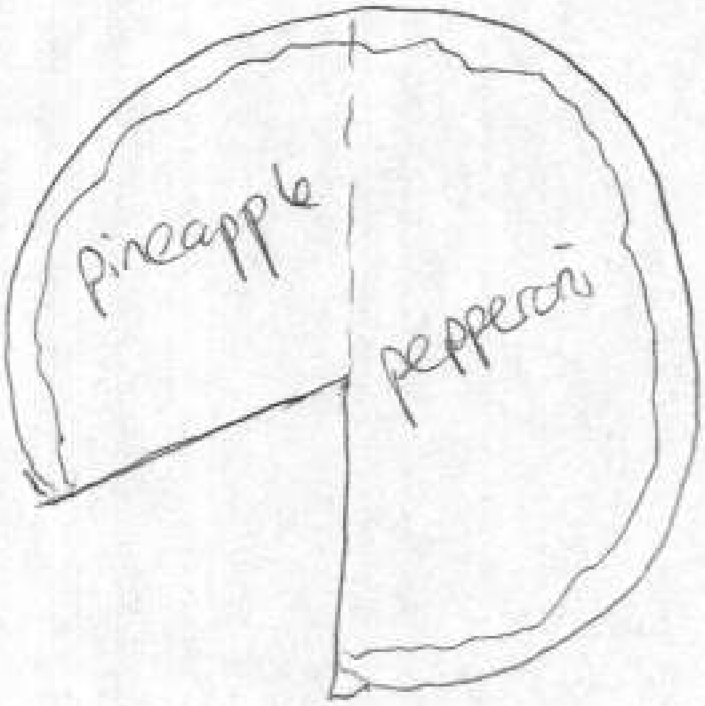
The next step is to pass out shares S_1, S_2, S_3 to the 3 players in a fair way if it can be done.

We see that in this situation it cannot be done — the only fair share among S_1, S_2, S_3 to both Clare and Charles is S_3, and they cannot both have it.

The solution is the following: Charles & Clare determine amongst themselves the least desirable share, and give it to Dan. In this case either S_1 or S_2 will do, as they are equally bad to Clare & Charles (but Dan sees each as $\frac{1}{3}$).

So they decide to give S_1 to Dan. Then Clare and Charles recombine S_2 and S_3 and use the divider-chooser method.

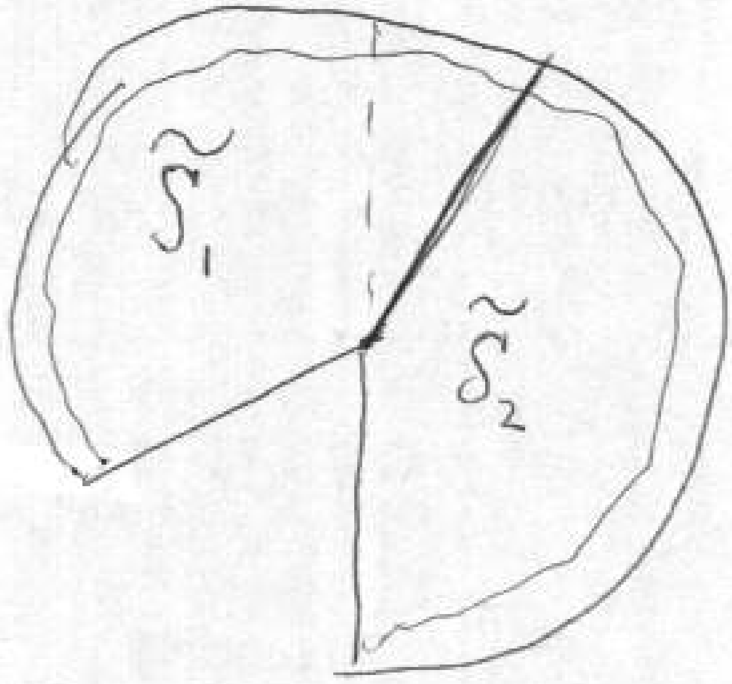
Suppose Clare is the new divider.



S_2 and S_3 combined

$= 7/9$ of physical area of original pizza

Clare divides this in a way fair to her, i.e. a division into equal parts:



$\tilde{S}_1 = \frac{5}{9}$ of the original pineapple half $+ \frac{2}{9}$ of the pepperoni half

$\tilde{S}_2 = \frac{7}{9}$ of the pepperoni half.

The worth of $\tilde{S}_1$ and $\tilde{S}_2$ to Clare & Charles:

	$\tilde{S}_1$	$\tilde{S}_2$
Clare	$1/2$	$1/2$
Charles	$2/9$	$7/9$

For example,

$$\text{value of } \tilde{S}_1 \text{ to Clare} = \frac{5/9 \times \left(\text{value of pineapple}\right) + \frac{2}{9} \times \left(\text{value of pepperoni}\right)}{\left(\text{total value of } S_2 + S_3\right)}$$

$$= \frac{5/9 \times \frac{1}{2} + 2/9 \times \frac{1}{2}}{7/9} = \frac{7/9 \times \frac{1}{2}}{7/9} = \frac{1}{2}$$

$$\text{value of } \tilde{S}_1 \text{ to Charles} = \frac{5/9 \times \left(\text{value of pineapple}\right) + \frac{2}{9} \times \left(\text{value of pepperoni}\right)}{\left(\text{total value of } S_2 + S_3\right)}$$

$$= \frac{5/9 \times 0 + 2/9 \times 1}{1} = \frac{2}{9}.$$