For a more interesting example we use the numbers from the text:

$$w_1 = 49 \quad \text{for} \quad P_1 = \text{republicans}$$

$$w_2 = 48 \quad \text{for} \quad P_2 = \text{democrats}$$

$$w_3 = 3 \quad \text{for} \quad P_3 = \text{independents}$$

The possibilities change:

"YES"	"No"	OUTCOME
$\{P_1\}$	$\{P_2, P_3\}$	NO
$\{P_2\}$	$\{P_1, P_3\}$	NO
$\{P_3\}$	$\{P_1, P_2\}$	NO
$\{P_1, P_2\}$	$\{P_3\}$	YES
$\{P_1, P_3\}$	$\{P_2\}$	YES
$\{P_2, P_3\}$	$\{P_1\}$	YES
$\{P_1, P_2, P_3\}$	$\{\}$	YES
$\{\}$	$\{P_1, P_2, P_3\}$	NO

Notation here: $[51; 49, 48, 3]$.

Definition: a <u>coalition</u> will be any set of players that decide to vote the same way. In the above example, $\{P_1\}$, $\{P_1, P_2\}$ are examples of coalitions

The <u>grand coalition</u> is the coalition consisting of <u>all</u> players. In the above example, $\{P_1, P_2, P_3\}$ is the grand coalition.

A <u>winning coalition</u> is a coalition that has enough votes to pass the motion, i.e. reach the quota.

The winning coalitions above in $[51: 49, 48, 3]$

are $\{P_1, P_2\}$, $\{P_1, P_3\}$, $\{P_2, P_3\}$

and $\{P_1, P_2, P_3\}$.

Note that the grand coalition is <u>always</u> a winning coalition.

A <u>critical player</u> is a player in a winning coalition such that if they are removed, the coalition is no longer a winning coalition.

Mathematically, a critical player is a player P of weight w in a winning coalition of total weight W if $W - w < q$.

The <u>critical count</u> of a player is the number of times that player is a critical player.

Let's look at the example $[51 : 49, 48, 3]$ above.

Winning Coalitions	Weight	Critical players
$\{P_1, P_2\}$	97	$\underline{P_1}$, $\underline{P_2}$
$\{P_2, P_3\}$	51	$\underline{P_2}$, $\underline{P_3}$
$\{P_1, P_3\}$	52	$\underline{P_1}$, $\underline{P_3}$
$\{P_1, P_2, P_3\}$	100	none

(underline means critical player)

Thus the critical counts are

B_1 = critical count for $P_1 = 2$

B_2 = —— " —— $P_2 = 2$

B_3 = —— " —— $P_3 = 2$

Banzhaf Power Index (BPI)

Consider a weighted voting system with players $P_1, P_2, \ldots, P_N$ and let $B_1, B_2, \ldots, B_N$ be their critical counts.

Set $T = B_1 + B_2 + \cdots + B_N$ = total critical count.

- The Banzhaf Power ~~Index~~ Index (BPI) of a player is the ratio of his/her critical count over the total critical count:

$$\beta_1 = \frac{B_1}{T}, \; \beta_2 = \frac{B_2}{T}, \; \ldots, \; \beta_N = \frac{B_N}{T}.$$

In the Senate example $[51: 49, 48, 3]$,

$B_1 = 2$, $B_2 = 2$, $B_3 = 2$, so $T = 2+2+2 = 6$.

Then $\beta_1 = \frac{B_1}{T} = \frac{2}{6} = \frac{1}{3}$ = BPI of P_1

$\beta_2 = \frac{1}{3}$ = BPI of P_2

$\beta_3 = \frac{1}{3}$ = BPI of P_3.

Thus in this example, republicans, democrats and independents all have the same Banzhaf Power Index.

One more bit of terminology:
we call the entire collection of BPI's, $\beta_1, \beta_2, \ldots, \beta_N$, the <u>Banzhaf Power Distribution</u> of the weighted voting system.

We can summarize how to compute the Banzhaf Power Distribution:

(8)

Step 1 List winning coalitions

Step 2 For each winning coalition determine the critical players

Step 3 Find the critical counts $B_1, \ldots, B_N$

Step 4 Find $T = B_1 + B_2 + \ldots + B_N$.

Step 5 Compute the BPI's $\beta_1 = \frac{B_1}{T}, \ldots, \beta_N = \frac{B_N}{T}$

Example: $[4 : 3, 2, 1]$.

Winning coalitions	weight	critical players
$\{\underline{P_1}, \underline{P_2}\}$	5	$\underline{P_1}, \underline{P_2}$
$\{\underline{P_1}, \underline{P_3}\}$	4	$\underline{P_1}, \underline{P_3}$
$\{\underline{P_1}, P_2, P_3\}$	6	$\underline{P_1}$

$B_1 = 3$, $B_2 = 1$, $B_3 = 1$, $T = 3 + 1 + 1 = 5$

$\beta_1 = \frac{B_1}{T} = \frac{3}{5}$, $\beta_2 = \frac{B_2}{T} = \frac{1}{5}$, $\beta_3 = \frac{B_3}{T} = \frac{1}{5}$.