Last time: Banzhaf Power.

Example: [4: 3, 2, 1]

We computed that the Banzhaf Power Distribution is given by $\beta_1 = \frac{3}{5}$, $\beta_2 = \frac{1}{5}$, $\beta_3 = \frac{1}{5}$.

In other words, this method of measuring power says that P_2 and P_3 have equal amounts of power, while P_1 has 3 times as much power as either one of P_2 or P_3.

We also looked at a U.S. Senate exaupk, [51: 49, 48, 3], and the Banzhaf Power Distribution is here equally divided: P_1, P_2, P_3 all have the same power.

Today we will learn about a different method to measure power: Shapley - Shubik Power.

(2) It will sometimes give the same answer as the Banzhaf Power does, but in general will be different.

The feature that distinguishes Shapley-Shubik Power from Banzhaf Power is the ordering of coalitions. This is not something we cared about before.

Let's look at the senate, $[51: 49, 48, 3]$.

Here are the "sequential coalitions":

$$\langle P_1, P_2, P_3 \rangle$$
$$\langle P_1, P_3, P_2 \rangle$$
$$\langle P_2, P_1, P_3 \rangle$$
$$\langle P_2, P_3, P_1 \rangle$$
$$\langle P_3, P_1, P_2 \rangle$$
$$\langle P_3, P_2, P_1 \rangle$$

The notation $\langle P_1, P_2, P_3 \rangle$ is similar to
$\{P_1, P_2, P_3\}$, but now <u>order matters</u>. So
$\langle P_1, P_2, P_3 \rangle$ means P_1 goes first, P_2 goes second, and P_3 goes last.

More formally, a <u>sequential coalition</u> is an *ordered* list of the players. The order of players is from left to right. The list contains <u>all</u> players.

In $[51 : 49, 48, 3]$ above, there were a total of 6 sequential coalitions. How many are there in a general weighted voting system with N players?

Let's count how many ways we can order the players $P_1, P_2, \ldots, P_N$. We first look at the possibilities of which player goes first. There are of course N possibilities. However, after a first player is chosen, there are now

④ only $N-1$ possibilities to be second place in line, since we can choose from all players, except the one that is already first. Then for third place we will have $N-2$ choices, and so forth.

Thus the number of sequential coalitions is equal to

$$N! \underset{\uparrow \atop \text{definition.}}{=} N \times (N-1) \times (N-2) \times (N-3) \times \cdots \times 2 \times 1.$$

In our example, of the senate, there were $3 = N$ players, and

$$3! = 3 \times 2 \times 1 = 6$$

was the number of sequential coalitions we counted.

The number $N!$ is called the <u>factorial of N</u> and appears everywhere in Mathematics (and in life!).