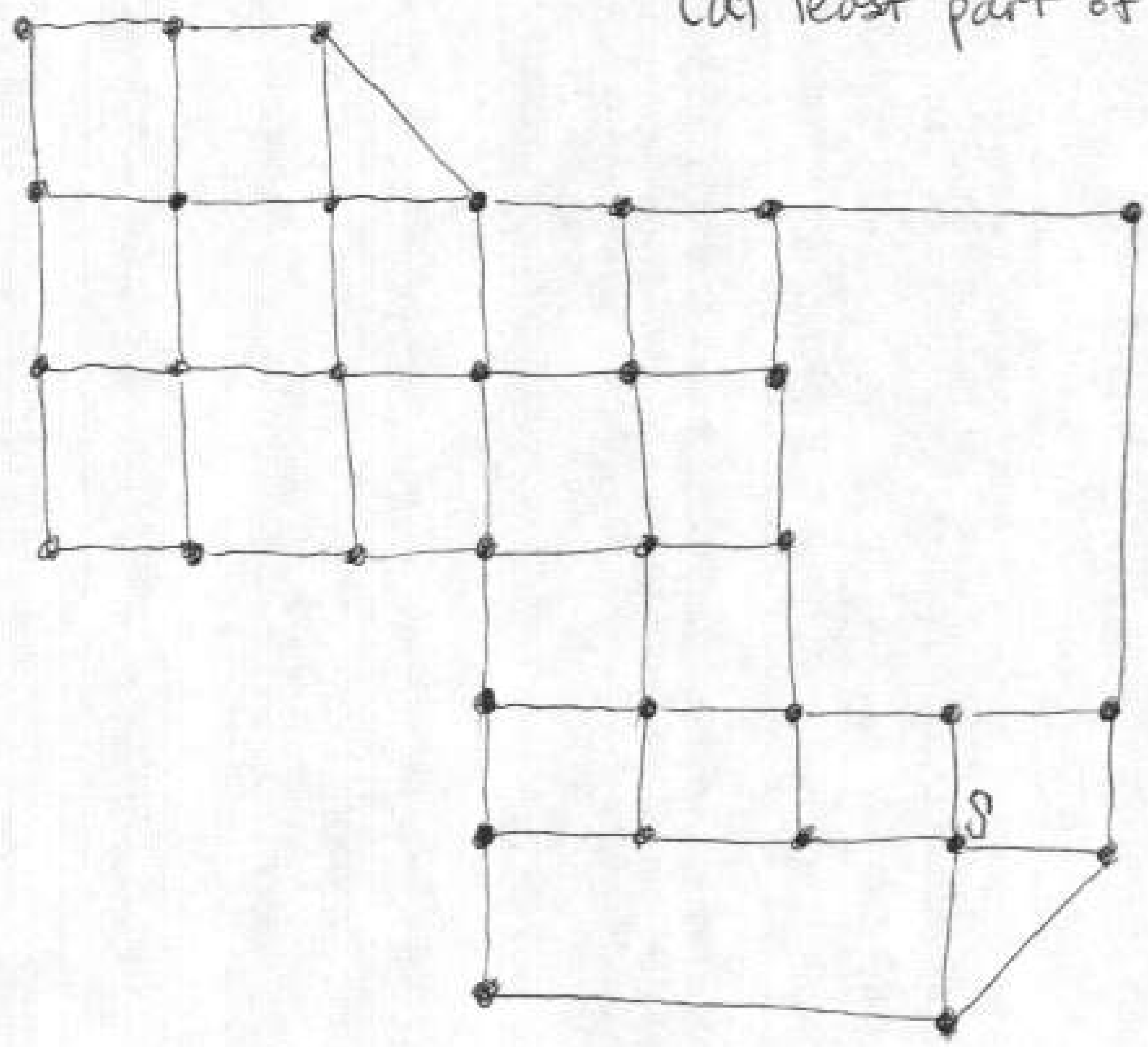
We begin by solving the security guard problem.
(at least part of it.)



Last time we phrased the problem in the language of graph theory: does the above graph have an Euler circuit? We will use:

Euler's Circuit Theorem:

- If a graph is connected and every vertex is even, then it has an Euler circuit
- If a graph has any odd vertices then it does not have an Euler circuit.

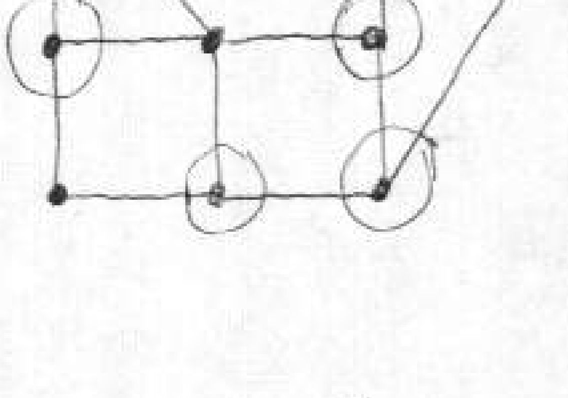
Otherwise said:

A graph has an Euler circuit if and only if it is connected and has only even vertices.

In the above graph it is easy to find some odd vertices, so the theorem tells us that there is no Euler circuit. Thus there is <u>no</u> way for the security guard to patrol the neighborhood covering every block exactly once.

Other examples:

(i)

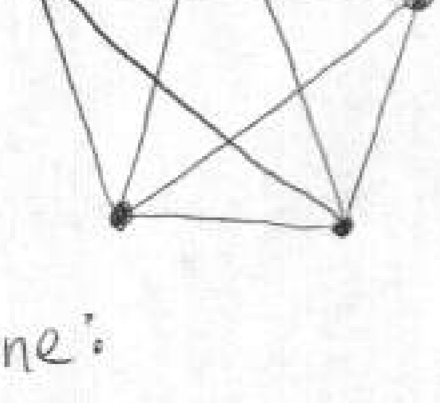


This graph has odd vertices (encircled) and so by Euler's circuit theorem it does not have an Euler circuit.

(ii)

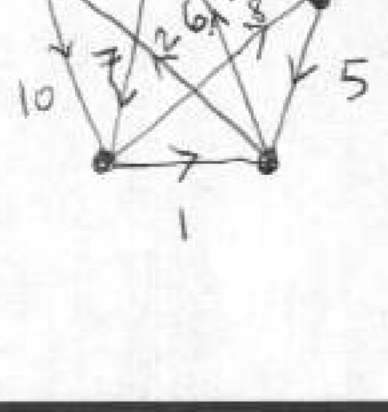
This graph has all even vertices, but it is not connected, so it doesn't have an Euler circuit.

(iii)

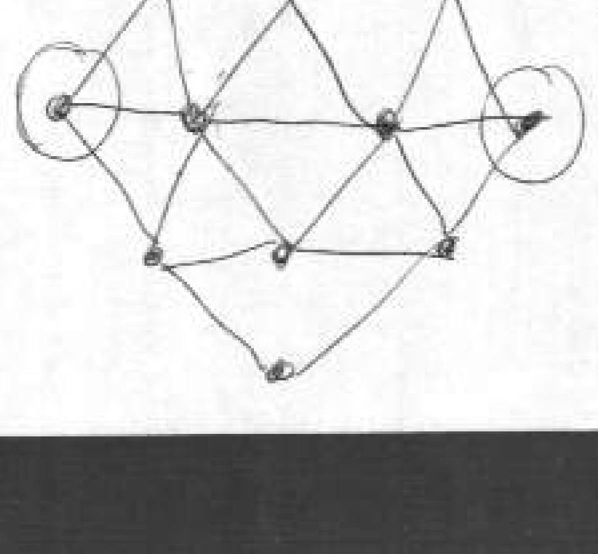


This graph (a 5-clique) is connected and has all even vertices, so it has an Euler circuit.

Here's one:



(iv)



Noj has two odd vertices

Remember another variation of the security guard problem: if there is no way for the guard to cover every block once and only once beginning and ending at the same location, is there a way for the guard to cover every block exactly once by beginning somewhere and ending at a possibly different location?

To answer this type of question we'll use the following result.

Euler's Path Theorem:

- If a graph is connected and has exactly two odd vertices, then it has an Euler path. Any such path begins and ends at the two odd vertices.
- If a graph has more than two odd vertices, then it has no Euler paths.

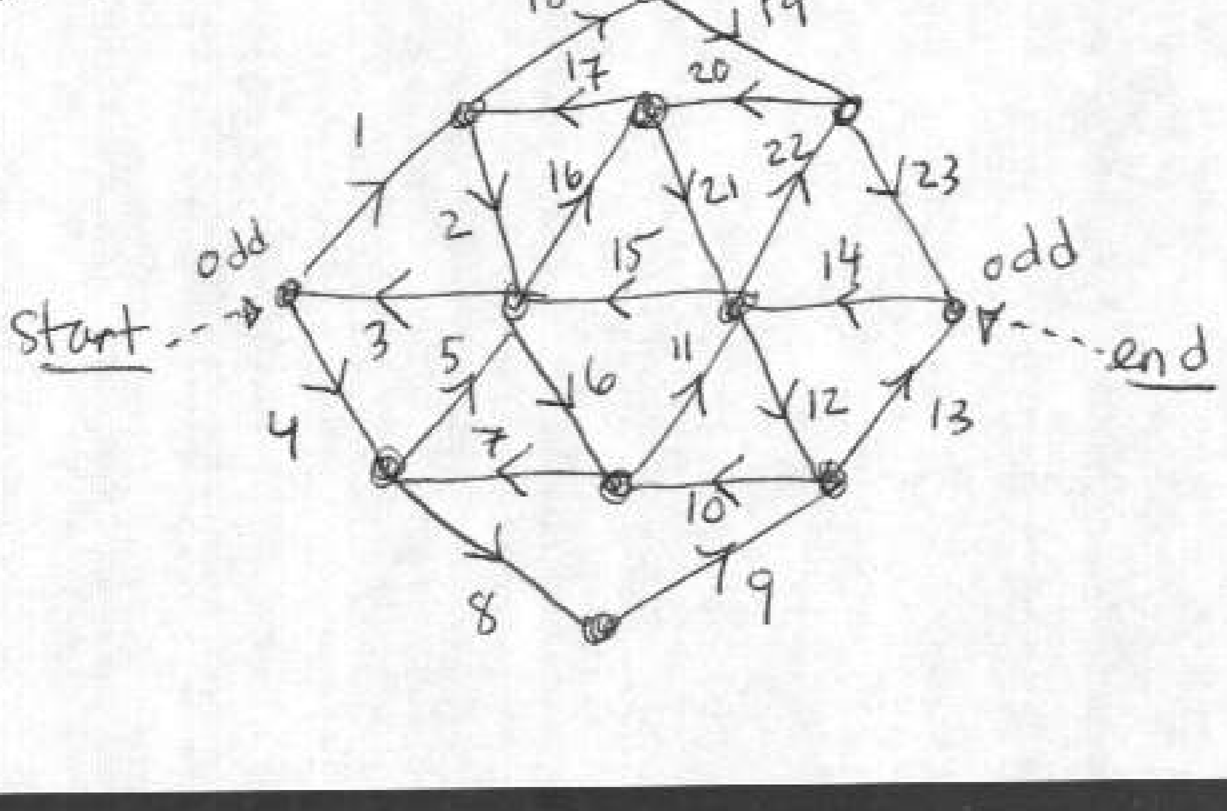
Referring to the graphs in our examples above, we have:

(i) has more than two odd vertices.
So this graph has <u>no</u> Euler paths.

(ii) this graph is not connected.
So it has <u>no</u> Euler paths.

(iii) this graph has an Euler circuit, which we may count as an Euler path. But it does <u>not</u> have any Euler paths beginning and ending in <u>different</u> locations; for this we need to have exactly two odd vertices.

(iv) this graph has no Euler circuits because it has two odd vertices. However, by the Theorem just introduced, we know it has an Euler path.
Here's one:



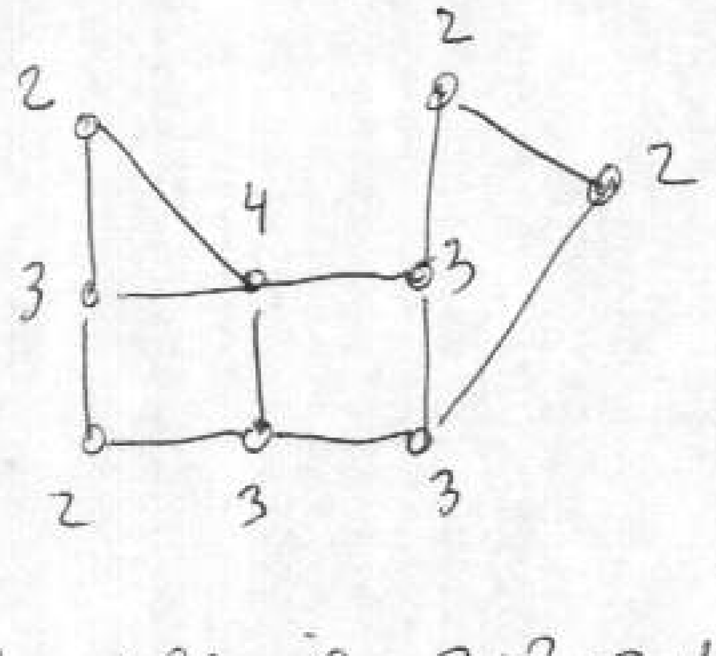
Since the security guard graph at the start had more than two odd vertices, we can say that it has no Euler paths, answering the second question we posed above in the negative.

Although not as essential to the questions we've been asking, we mention another insightful theorem:

Euler's Sum of Degrees Theorem:

$$\left(\begin{array}{c}\text{Sum of}\\ \text{degrees of a}\\ \text{graph}\end{array}\right) = 2 \times \left(\begin{array}{c}\text{\# edges in}\\ \text{the graph}\end{array}\right)$$

For example, in graph (i) above, the degrees are 2, 3, 2, 4, 3, 3, 3, 2, 2:



The sum of degrees is $2+3+2+4+3+3+3+2+2 = 24$.

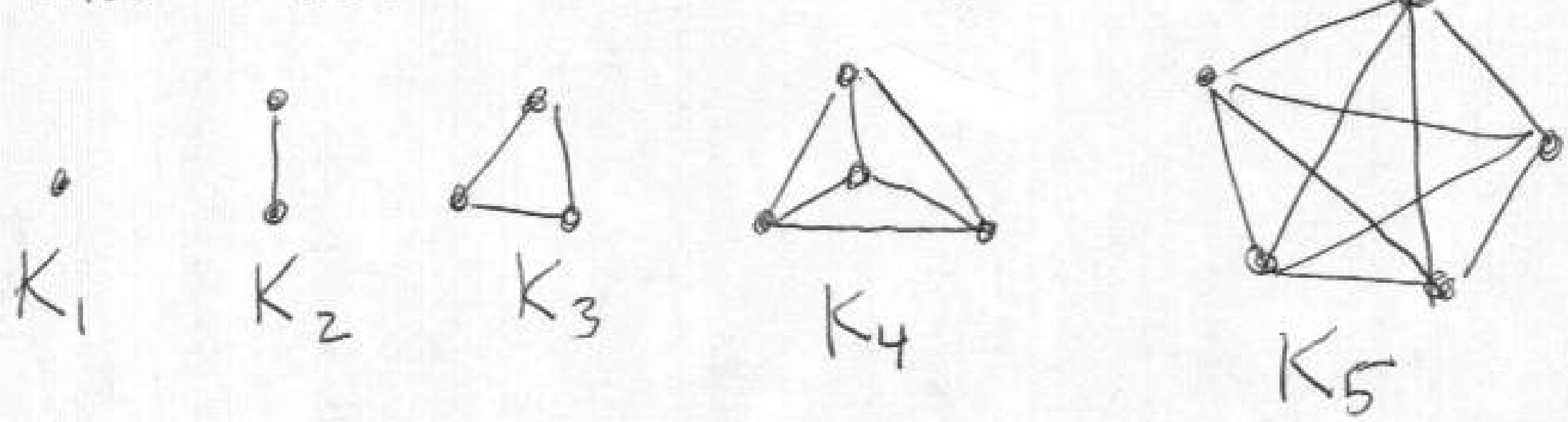
The theorem then says

$$24 = \begin{pmatrix}\text{sum of}\\ \text{degrees}\end{pmatrix} = 2 \times (\#\,\text{edges})$$

so that #edges = 12. We look at the graph, and lo and behold, it has 12 edges!

Here's an application of the theorem.

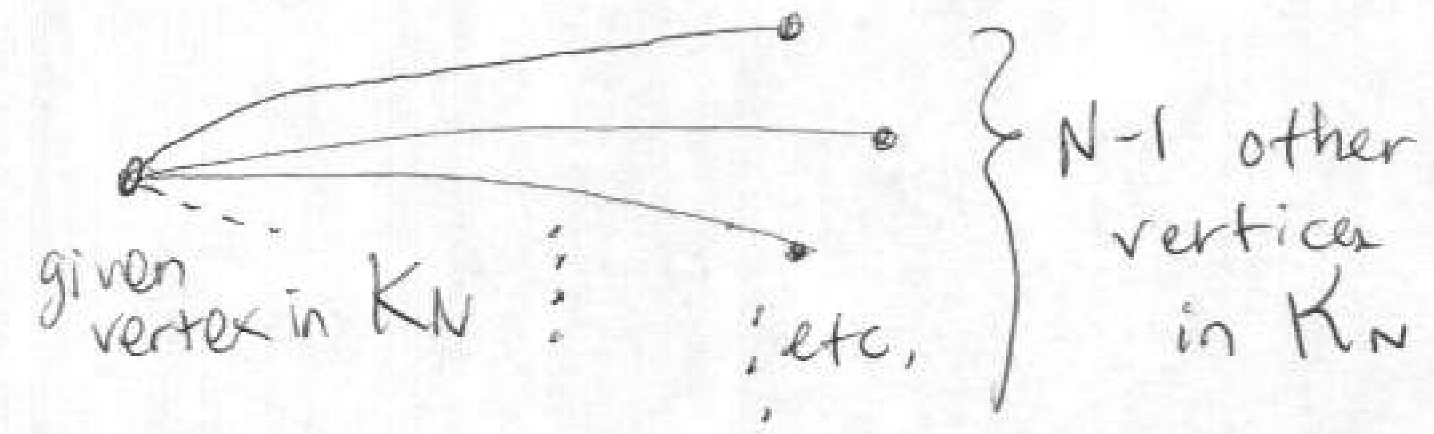
Consider the complete graph K_N with N vertices. (We also called this the N-clique in class.)



The graph K_N has exact one edge between every two vertices, and has N vertices total.

The degree of any given vertex is $N-1$, since the vertex has exactly one edge going to each of the other $N-1$ vertices.

i.e.,



Since there are N vertices total, the sum of degrees is given by

$$\begin{pmatrix}\text{sum of}\\ \text{degrees}\end{pmatrix} = (N-1) + (N-1) + \dots + (N-1) = N \times (N-1)$$

degree of each vertex; there are N terms total

The theorem then says

$$N \times (N-1) = 2 \times (\#\,\text{edges})$$

So we obtain the following formula:

$$\begin{pmatrix}\#\,\text{edges}\\ \text{in } K_N\end{pmatrix} = \frac{N \times (N-1)}{2}.$$

For example, look at K_5 above (the pentagram). You can count that the # edges is 10. However you could have also used the formula: $\frac{5 \times (5-1)}{2} = \frac{5 \times 4}{2} = \frac{20}{2} = 1$