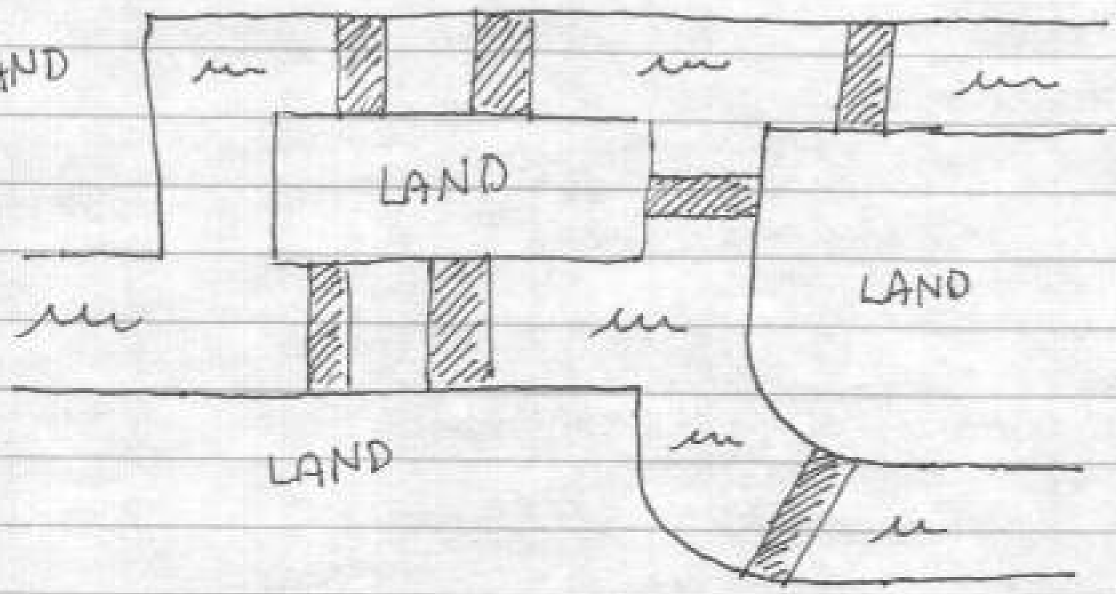
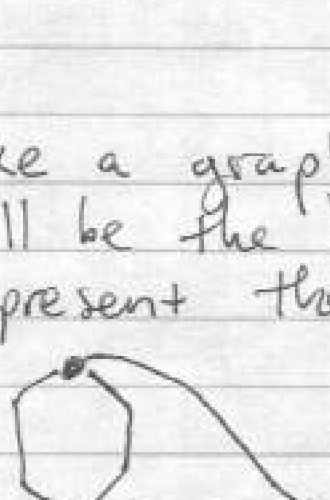
Graph Theory (Cont.): Review of last time / Fleury's Algorithm

The Bridges of Königsberg Puzzle:



In the 1700's in Königsberg (now Kaliningrad, Russia) it was a game for locals to try and walk around the city crossing every one of the seven bridges exactly once. It was believed this was impossible and Euler was asked to rigorously prove this. In doing so he invented graph theory.

To begin, make a graph of the above map; the vertices will be the land masses and the edges will represent the bridges:



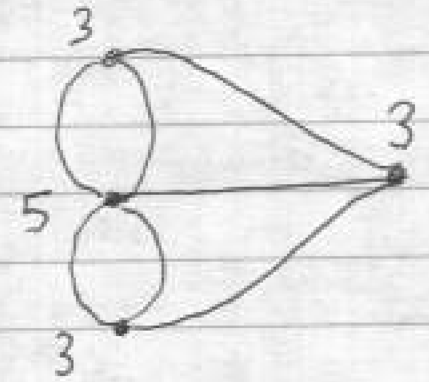
Now the Königsberg puzzle is: does this graph have an Euler circuit or an Euler path?

Recall from last time:

Euler's Circuit Thm:

- if a graph is connected and every vertex is even, then it has an Euler Circuit.
- if a graph has any odd vertices, then it does not have an Euler Circuit.

Here is our graph with degrees of vertices written in:



All four vertices are odd, so by the Theorem, there is no Euler Circuit. Recall also:

Euler's Path Theorem:

- if a graph is connected and has exactly two odd vertices, then it has an Euler path.
- if a graph has more than two odd vertices, then it has no Euler path.

We see then that our graph also has no Euler paths.

When the above theorems tell us that there is an Euler path or Euler circuit, we are not told how to find such paths. For this we use:

Fleury's Algorithm for finding an Euler Circuit (or Path)

Step 0: Make sure the graph is connected and either ① has no odd vertices (in which case it has an Euler circuit) or ② h[illegible] just two odd vertices (in which case it has an Euler path).

Step 1: Choose a starting vertex.

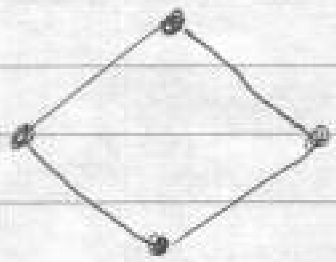
In case ① this can be any vertex;
in case ② this must be one of the two odd vertices

Step 2: At each step, if you have a choice, don't choose a bridge of the yet-to-be-travelled part of the graph.
If you don't have a choice, just take the one route available.

A bridge is an edge of a connected graph that if removed makes the graph disconnected.

Examples:

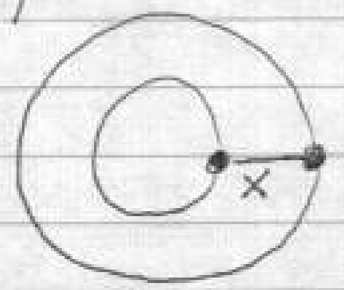
(i)



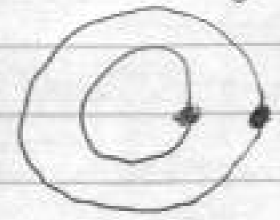
if any edge is removed, this graph is still connected.

So it has no bridges.

(ii)



if we remove edge X then the result is

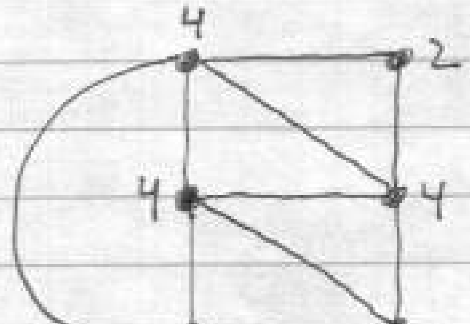


which is disconnected.

So edge X is a bridge.

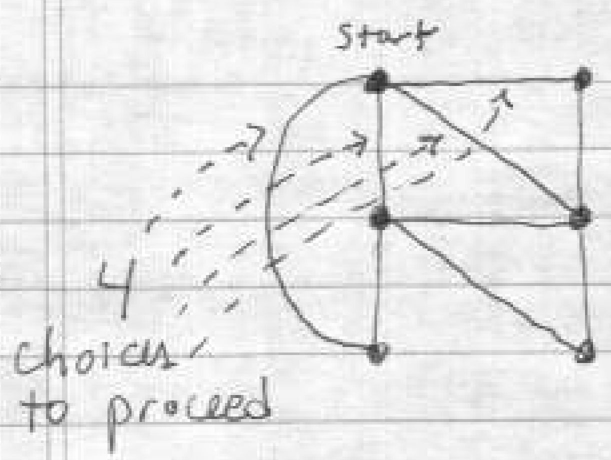
After repeating Step 2 of the algorithm we eventually finish the circuit or path (the text calls this Step 3).

Let's apply the algorithm to the graph



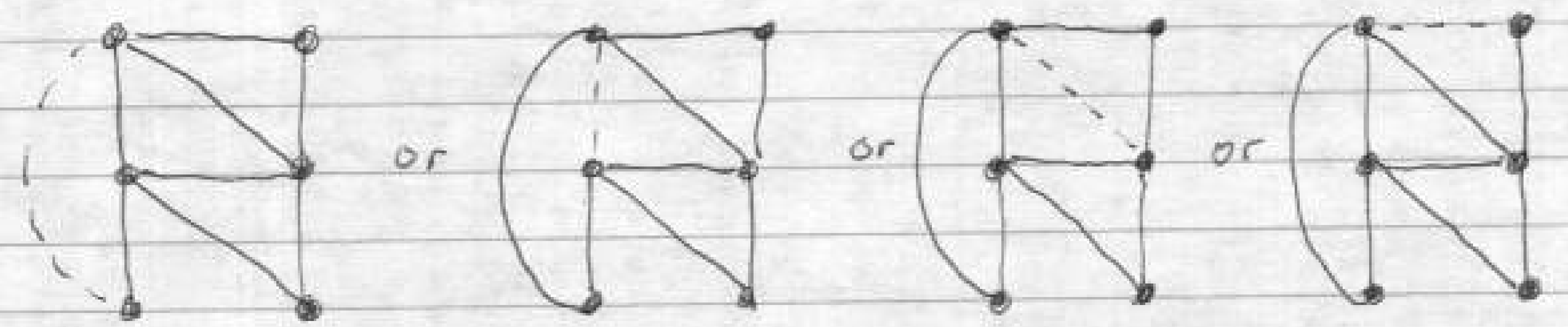
Step 0: The degrees of the vertices are written, and we see that the graph has only even vertices, and is connected, so it has an Euler circuit.

Step 1: We then choose any vertex to start our circuit. Let's choose the top left vertex of degree 4.



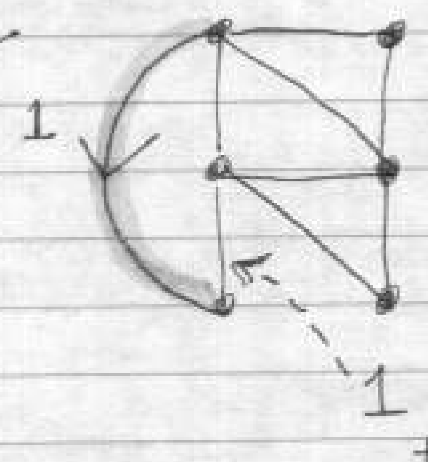
Step 2 We have 4 directions from our starting vertex to choose from.

If each of these edges are removed, the result (of the yet-to-be-travelled graph) is



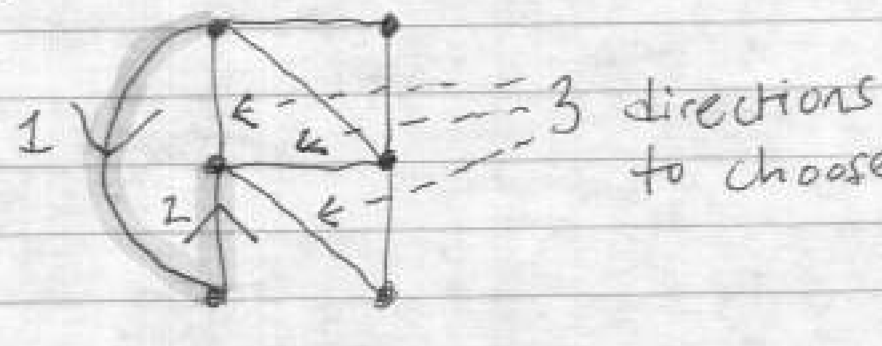
So none of these edges are bridges — all these are connected. Thus we can choose any of the 4 directions to proceed. We choose the left one.

Step 2 again

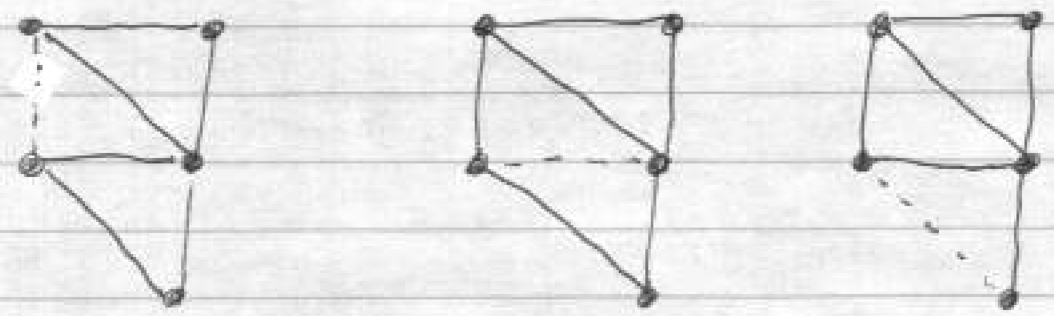


Next, we see we only have one direction to choose from, so we take it.

Step 2 again

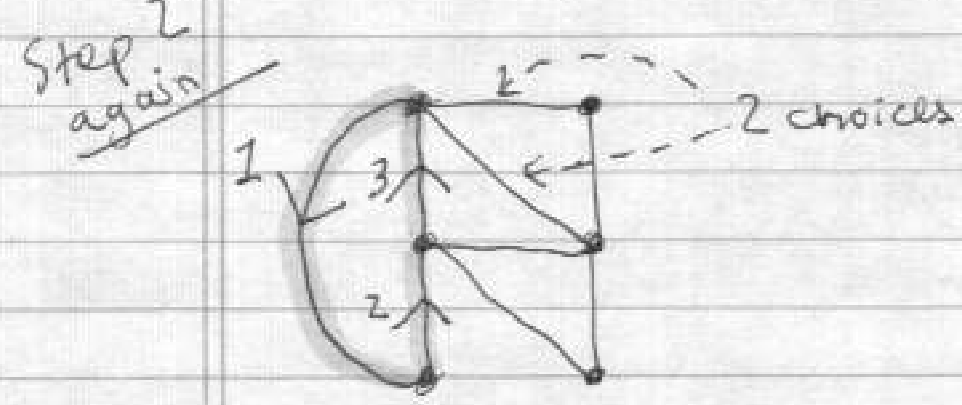


If each edge choice is removed from the yet-to-be travelled graph the results are

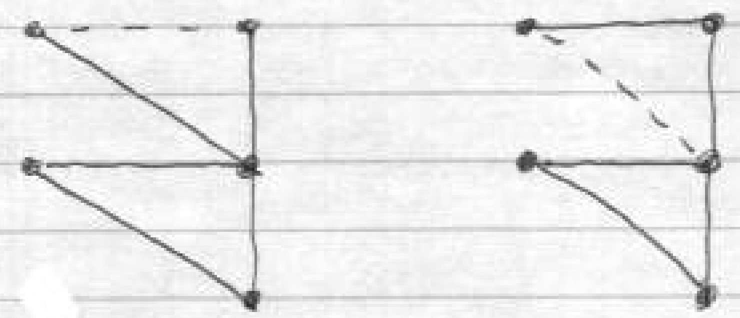


each is connected, so none of the 3 edges are bridges of the yet-to-be-travelled graph. We can go for any of them.

We choose the top-left one:



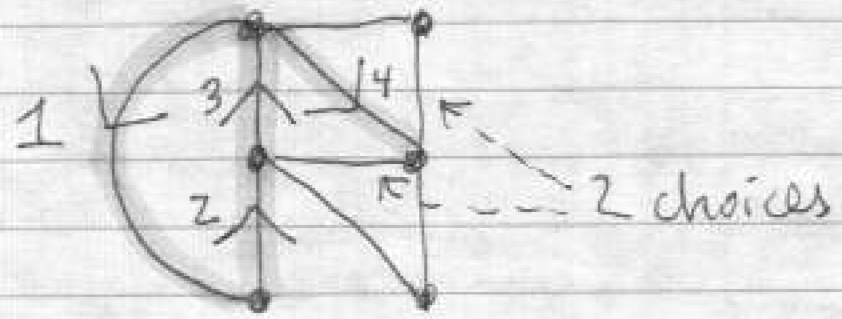
If we remove these edges from the yet-to-be-travelled graph we obtain



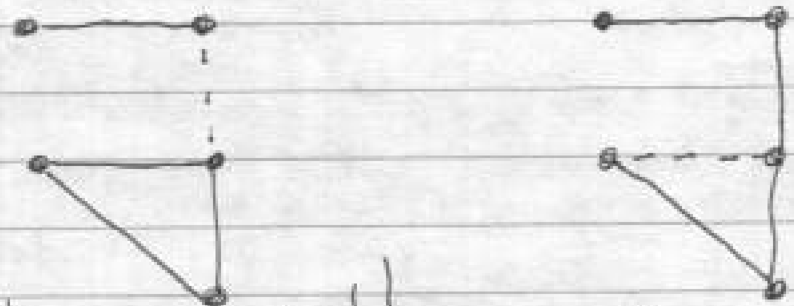
Since both are connected, either direction choice is okay.

We choose the bottom one (corresponding the above right-hand picture).

Step 2 again



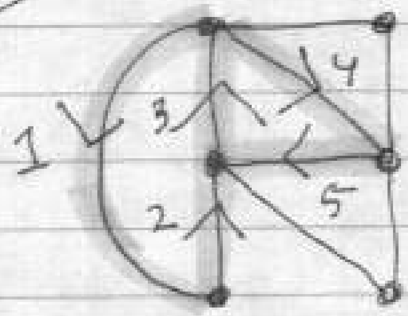
If we remove the 2 choices from the yet-to-be-travelled graph then:



disconnected! connected

The vertical edge choice <u>is</u> a bridge of the yet-to-be travelled graph, so is not a valid direction, according to Fleury's algorithm. We must take the horizontal direction, and we do that.

Step 2 again



The rest of the steps involve only one direction choice, and the finished circuit is: