Ch. 9: Population Growth Models.

A sequence is an infinite, ordered list of numbers.

Examples:

(1) 1, 1, 1, 1, 1, 1, 1, 1, ...

(2) 1, 2, 3, 4, 5, 6, 7, 8, ...

(3) 3, 5, 7, 9, 11, 13, 15, ...

(4) 1, 11, 21, 1211, 111221, 312211, 13112221, ...

We often write sequences in symbols as

$$A_1, A_2, A_3, A_4, \ldots$$

Often there is an explicit formula for a given sequence; for any $N \geq 1$ this lets one plug N into an equation to obtain A_N. Some of the examples above have formulas

(1) $A_N = 1$

(2) $A_N = N$

(3) $A_N = 2N+1$

Example (4) is much more interesting.

$A_1 = 1$

A_2 = say A_1 aloud = "one 1" = 11

A_3 = say A_2 aloud = "two 1's" = 21

A_4 = say A_3 aloud = "one 2, one 1" = 1211

etc.

We don't have a formula for A_N, but we do have a recipe for getting A_N from the previous entry, A_{N-1}. This is an example of <u>recursion</u>.

<u>Example</u>: 1, 2, 6, 24, 120, 720, ...

This has an explicit formula: $A_N = N!$

But it also has a nice recursive formula:

$$A_1 = 1, \qquad A_N = N \times A_{N-1}.$$

<u>Example</u>: The sequence with the recursive formula

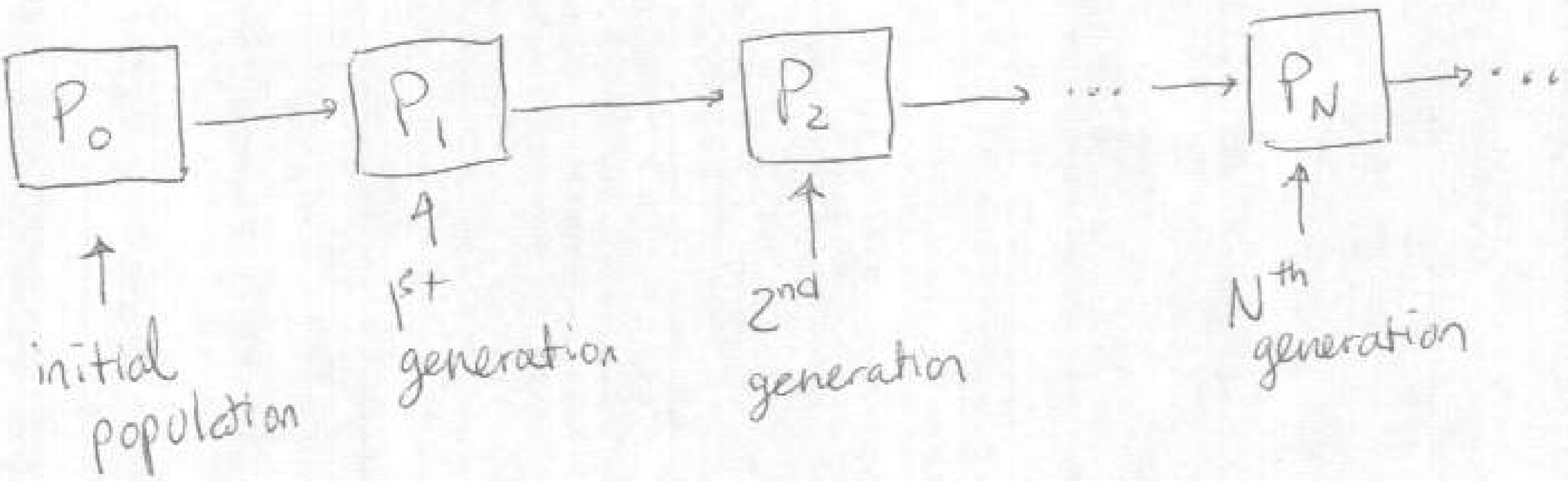
$A_N = A_{N-1} + 1$ and $A_1 = 1$ is example (2)

A <u>population sequence</u> describes the size of a population as it changes over time, measured in discrete time intervals.

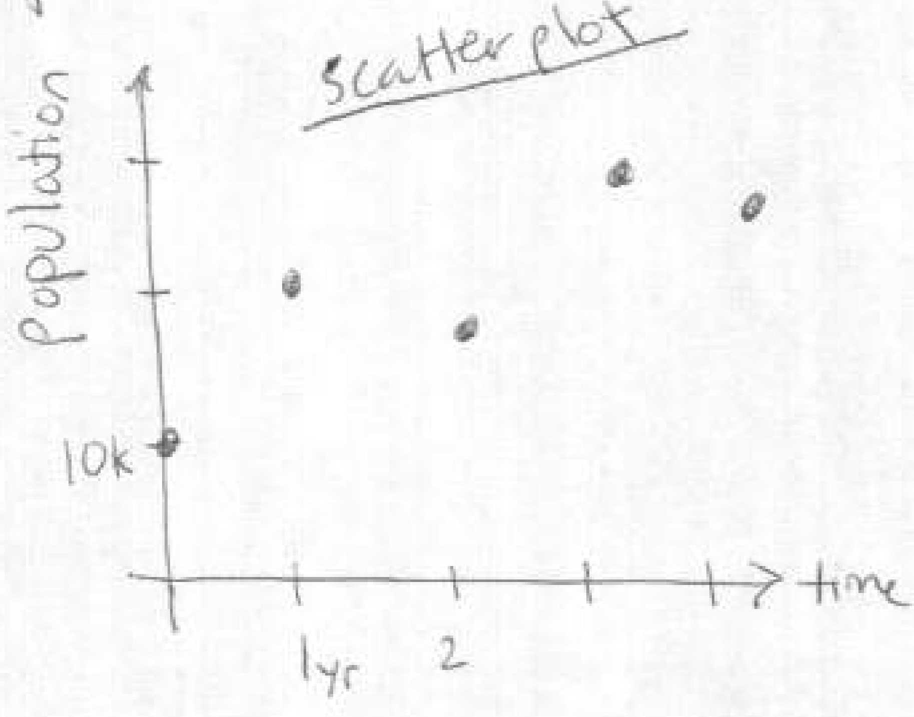
We usually denote such a sequence by

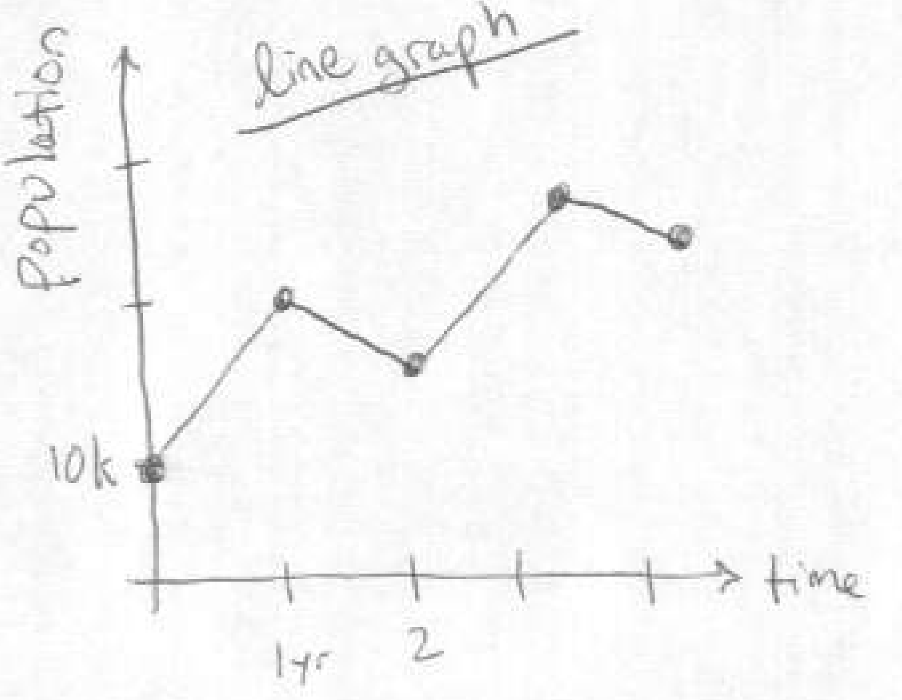
$$P_0, P_1, P_2, P_3, \ldots$$

and give these populations names:



We often depict a population sequence using a <u>time-series graph</u>:





Example: Fibonacci's Rabbits

Consider the following situation: we start with 1 pair of rabbits. After 1 month, this pair is mature enough to reproduce and create another pair by the third month. Specifically:

$P_0 = 1$ pair (not mature)

$P_1 = 1$ pair (now mature)

$P_2 = 2$ pairs (the mature pair, plus the new pair)

This process continues: the mature pair produces another pair, but the new pair is not mature enough yet, so:

$P_3 = 3$ pairs (original mature pair, the pair that was not mature but now is, and the new pair)

To continue it is more convenient to keep track of which pairs are mature:

mature	0	1	1	2	3	5
		add		add	add	add
new	1	0	1	1	2	3
	$P_0=1$	$P_1=1$	$P_2=2$	$P_3=3$	$P_4=5$	$P_5=8$

(P_N = sum of two numbers right above)

The number $F_N = P_{N-1}$ is called the N^{th} Fibonacci number. It is not hard to check that F_N satisfies the recursion

$$F_N = F_{N-1} + F_{N-2}. \quad (\star)$$

There is actually a closed formula for F_N, which we will currently derive.

First we hypothesize that there are positive numbers k and x such that

$$F_N = kx^N.$$

Then $(\star)$ implies

$$kx^N = kx^{N-1} + kx^{N-2}$$

and dividing both sides by kx^{N-2} we get

$$x^2 = x + 1$$

or, rearranging, we have

$$x^2 - x - 1 = 0.$$

Recall the quadratic formula:

if $ax^2+bx+c=0$ then $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

For $x^2-x-1=0$ we have $a=1, b=-1, c=-1$. So we get

$$x=\frac{1\pm\sqrt{1-4\times(-1)}}{2}=\frac{1\pm\sqrt{5}}{2}.$$

There are thus two possibilities for x.
We ammend our original hypothesis to supposing that F_N is a linear combination of the two possibilities for x^N:

$$F_N=k_1\left(\frac{1+\sqrt{5}}{2}\right)^N+k_2\left(\frac{1-\sqrt{5}}{2}\right)^N \quad (**)$$

where k_1, k_2 are constants to be solved.
Now we did not define F_0, but we may as well set $F_0=0$. Then

$$0=F_0=k_1\left(\frac{1+\sqrt{5}}{2}\right)^0+k_2\left(\frac{1-\sqrt{5}}{2}\right)^0=k_1+k_2$$

implies that $k_1=-k_2$.

Next, since $F_1 = 1$ we have

$$1 = F_1 = k_1\left(\frac{1+\sqrt{5}}{2}\right)^1 - k_1\left(\frac{1-\sqrt{5}}{2}\right)^1$$
$$= k_1\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)$$
$$= k_1\left(\sqrt{5}\right)$$

so that $k_1 = \frac{1}{\sqrt{5}}$.

Plugging into ($\ast\ast$) yields

$$F_N = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^N - \left(\frac{1-\sqrt{5}}{2}\right)^N\right)$$

a wonderful closed formula for F_N!