Ch. 9 Continued (Linear Growth, Arithmetic Sequences)

Recall that we were studying sequences, and in particular population growth sequences.

Example: Suppose we are told that a sequence

$$A_1, A_2, A_3, A_4, \ldots$$

is defined by $A_N = \#(\text{ways of choosing two out of } N \text{ objects})$ so that $A_1 = 0$ (can't choose two objects from a collection of one object) and $A_2 = 1$ (there's one way to choose two things from two things — just choose both!).

We can continue to find that

$$A_1 = 0, \; A_2 = 1, \; A_3 = 3, \; A_4 = 6, \; A_5 = 10, \ldots$$

but we can also find an explicit formula:

$$A_N = \#(\text{ways of choosing } 1^{st} \text{ object}) \times \#(\text{ways of choosing } 2^{nd}) / 2$$

we divide by 2 b/c there are 2 ways of ordering each pair of choices, and we are double counting those, b/c we don't care about order.

Then $A_N = N(N-1)/2$.

We can check

$A_1 = 1(1-1)/2 = 0$

$A_2 = 2(2-1)/2 = 1$

$A_3 = 3(3-1)/2 = 3$

$A_4 = 4(4-1)/2 = 6$

$A_5 = 5(5-1)/2 = 10$, etc.

just to make sure it agrees with our counting.

The number $\frac{N(N-1)}{2}$ is often written $\binom{N}{2}$

and is called "N choose 2".

Example: Let $A_N = 1+2+3+\dots+N$.

Then $A_1 = 1$

$A_2 = 1+2 = 3$

$A_3 = 1+2+3 = 6$

$A_4 = 1+2+3+4 = 10$

It looks like A_N should equal $\frac{N(N+1)}{2} = \binom{N+1}{2}$.

We can prove this, using clever observation due the Mathematician Gauss, when he was a child (although the formula goes back to the Greeks):

$$\begin{array}{ccc} A_N & & 1+2+3+4+\cdots+N \\ + & = & + \\ A_N & & N+(N-1)+(N-2)+(N-3)+\cdots+1 \\ \parallel & & \parallel \\ 2A_N & & \underbrace{(1+N)+(1+N)+\cdots+(1+N)}_{N \text{ times}} = N(N+1) \end{array}$$

So $2A_N = N(N+1)$, which then gives $A_N = N(N+1)/2$.

A_N here are also sometimes called the <u>triangle numbers</u> since the count the # of points in a triangle made of dots of height N:

etc.

$A_1 = 1$ $\quad A_2 = 3$ $\quad A_3 = 6$ $\quad A_4 = 10$

Now let's return to population growth sequences.

Last time we considered Fibonacci's Rabbit scenario, and his sequence of numbers, which we also found an explicit formula for.

Now we consider some other more standard models of population growth.

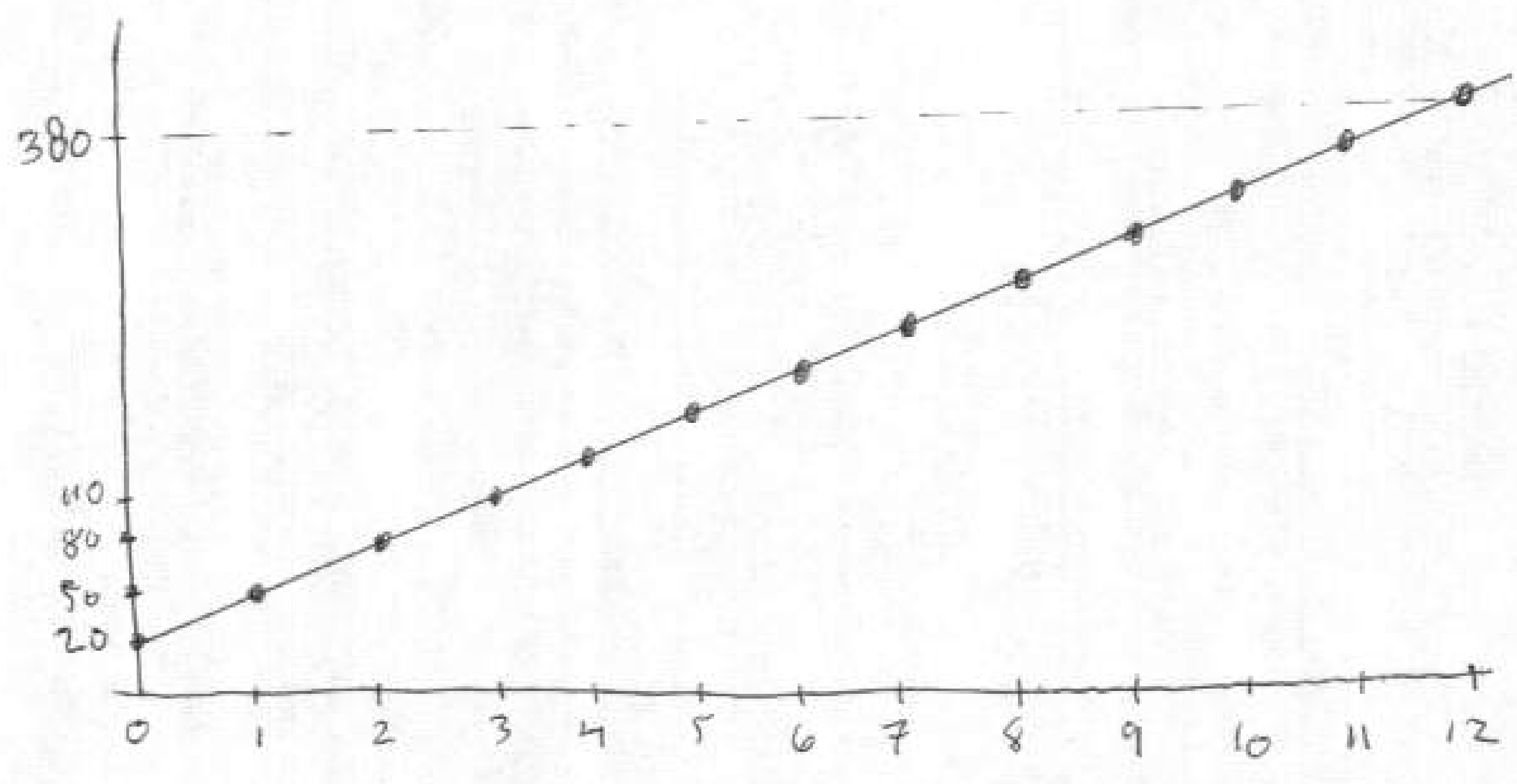
The Linear Growth Model

A population has <u>linear growth</u> if each generation grows by a constant amount.

<u>Example:</u> A new chain of restaurants is opening more and more locations as business increases. In January they had 20 locations, but each month they've been adding 30 stores. How many do they have by the end of the year?

$$\boxed{P_0 = 20} \xrightarrow{+30} \boxed{P_1 = 50} \xrightarrow{+30} \boxed{P_2 = 80} \xrightarrow{+30} \boxed{P_3 = 110} \longrightarrow \cdots$$

Jan. Feb. Mar. April



We see that $P_{12} = 20 + 12 \times 30 = 380$

So by the end of the year they have 380 stores.

(The growth is linear because on a graph, it forms a line.)

In general, we have some initial population P_0 that changes each generation by some fixed amount d (above, $d=30$). Then the population sequence is

$$P_0, \quad P_0 + d, \quad P_0 + 2d, \quad P_0 + 3d, \quad P_0 + 4d, \ldots$$

so that an explicit formula for P_N is given by

$$P_N = P_0 + Nd.$$

Note that we have a recursive formula $P_N = P_{N-1} + d$ also

This kind of sequence (abstractly, and not necessarily to do with population growth) is called an arithmetic sequence.

Example: The sequence A_N defined by $A_N = N$ is arithmetic. Indeed, $A_N = A_{N-1} + 1$, so here $d = 1$, and $A_1 = 1, A_2 = 2, A_3 = 3 \ldots$

<u>Example</u>: The sequence $A_N = 1+2+\dots+N$ defined earlier is <u>not</u> arithmetic: $A_N = N + A_{N-1}$, so the difference between two terms in the sequence is <u>not</u> constant (it is N).

<u>The Arithmetic Sum Formula:</u>

If $P_0, P_1, P_2, \dots, P_{N-1}$ are terms in an arithmetic sequence, then

$$P_0 + P_1 + P_2 + \dots + P_{N-1} = \frac{(P_0 + P_{N-1})N}{2}.$$

This generalizes the formula

$$1+2+3+\dots+N = \frac{N(N+1)}{2}$$

we found earlier.

<u>Example</u>: Consider the arithmetic sequence $5, 8, 11, 14, 17, \dots$. Here $P_0 = 5$ and $d = 3$. Find the sum of the first 500 terms.

The 500th term is

$$P_{499} = P_0 + 499 \times d = 5 + 499 \times 3$$
$$= 5 + 1497 = 1502.$$

Then the sum of the first 500 terms is

$$P_0 + \cdots + P_{499} = \frac{(P_0 + P_{499}) \times 500}{2}$$
$$= \frac{(5 + 1502) \times 500}{2}$$
$$= 1507 \times 500/2$$
$$= 376,750.$$