Ch. 9 (continued): More linear growth, Exponential growth.

We start by reviewing what we learned about linear growth.

Example: A population of Rabbits is growing linearly. The initial population is 10, and the 20th generation is of size 510. How many rabbits are born each new generation?

Recall $P_N = P_0 + N \cdot d$. Here $P_0 = 10$ and we are told $P_{20} = 510$, and we must solve for d.

$$P_{20} = P_0 + 20 \times d$$

$$\Rightarrow 510 = 10 + 20d$$

$$\Rightarrow 500 = 20d$$

$$\Rightarrow d = 500/20 = 25.$$

So the answer is 25.

Example: Is the sequence 1, 5, 9, 13, 18, ... arithmetic?

To be arithmetic, the difference between any two consecutive terms should be constant.

We find that $5-1=4$

$$9-5=4$$
$$13-9=4$$
$$18-13=5$$

Since the difference $18-13$ is different, the sequence is not arithmetic.

Example: The net profits a company is making, per month, is increasing linearly (each month

In January of 2015 they made \$50K, while in January of 2017 they made \$650k.

(a) How much are profits increasing each month?

(b) How much money has the company made from Jan, 2015 to Jan 2017?

(including profits from Jan 2015)

(a) We have $P_0 = 50k$ and $P_{24} = 650k$.

On the other hand, $P_{24} = P_0 + 24 \times d$, so

$$650 = 50 + 24d$$
$$\Rightarrow 600 = 24d$$
$$\Rightarrow d = 600/24 = 25.$$

So profits are increasing $\boxed{\$25k \text{ per month.}}$

(b) We don't even need part (a) to solve this part. Recall the arithmetic sum formula:

$$P_0 + P_1 + P_2 + \dots + P_{N-1} = \frac{(P_0 + P_{N-1})N}{2}$$

In our situation this gives $(N=25)$

$$P_0 + P_1 + \dots + P_{24} = \frac{(50 + 650) \times (25)}{2}$$

$$= 8750.$$

So over this time period the company has made 8750k, or $\boxed{\$8,750,000}$.

<u>Exponential Growth</u>

Classic illustration:

you recently won a monetary prize, and you have 2 options for receiving it:

(1) get \$100,000 right now

(2) get 1 penny today, 2 pennies tomorrow, 4 pennies the next day, etc. for a month (31 days).

Consider a sequence $A_1, A_2, A_3, A_4, \ldots$

If this sequence satisfies the recursion

$$A_N = RA_{N-1}$$

for some constant R, we say that we have a <u>geometric sequence</u>.

Note $A_N = RA_{N-1} = R*RA_{N-2} = R^2A_{N-2} = R^3A_{N-3}$
$= \ldots = R^{N-1}A_1$.

So an explicit formula for a geometric sequence is given by

$$A_N = R^{N-1}A_1.$$

<u>Example</u>:

Consider option #2 above, involving the pennies.

$A_1 = 1$ penny
$A_2 = 2$ pennies
$A_3 = 4$ pennies ... and they keep doubling,

i.e. $A_N = 2A_{N-1}$. So $A_N = 2^{N-1} \cdot A_1 = 2^{N-1}$.

On day 31, we get $A_{31} = 2^{30} = 1{,}073{,}741{,}824$ pennies, or \$10,737,418.24!

We get over \$10 million on day 31, and this is not even counting what we got from previous days!

(Later we'll compute the total amount we get.)

Now let's turn to populations.

The growth rate $\underline{\underline{r}}$ of a population as it changes from an initial value X to a new value Y is defined to be

$$r = \frac{Y-X}{X}.$$

A population sequence $P_0, P_1, P_2 \ldots$ grows exponentially if the sequence is a geometric sequence, i.e. $P_1 = RP_0$, $P_2 = R^2P_0$, $P_3 = R^3P_0$ for some R, a constant.

Equivalently, a population sequence grows exponentially if the growth rate from one generation to the next is constant.

If the growth rate is r, the relation to R is given by:

$$r = \frac{P_1 - P_0}{P_0} = \frac{RP_0 - P_0}{P_0} = R - 1,$$

so $R = 1 + r$, and we can also write

$$P_0,\ (1+r)P_0,\ (1+r)^2 P_0,\ (1+r)^3 P_0, \ldots$$

for our exponential growth sequence.

Example: An epidemic is spreading exponentially. The initial population infected ("population zero") is 1 person, and so $P_0 = 1$.

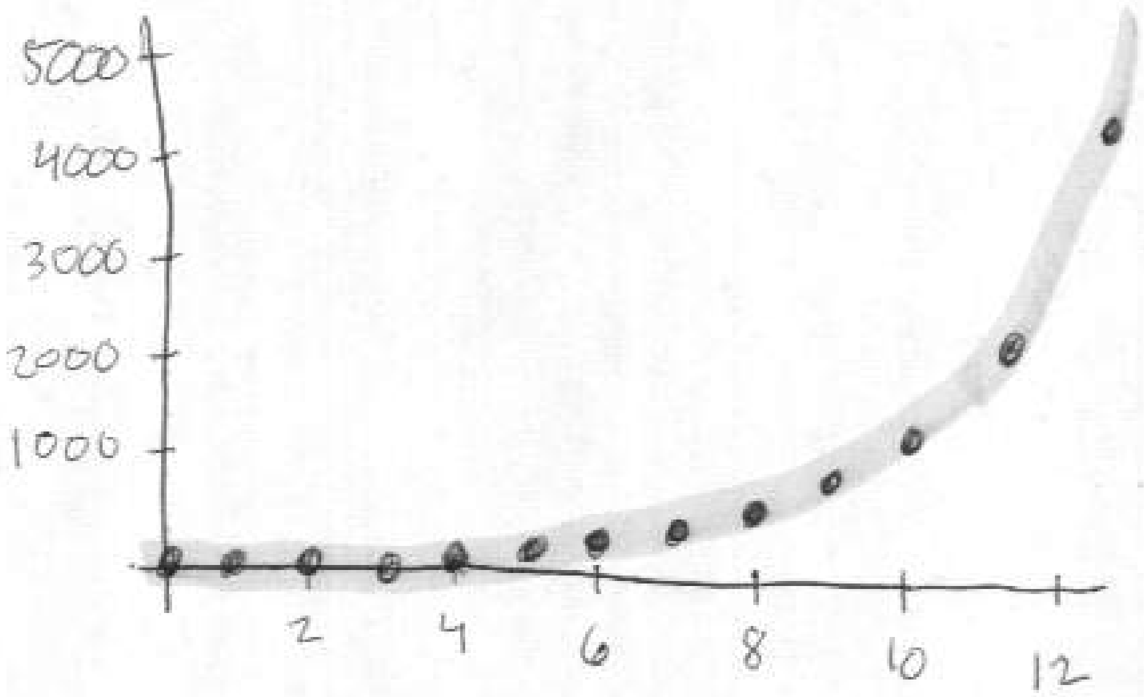
We are told that $P_{12} = 4096$, i.e. 4,096 people are infected after 12 months.
What is the rate of growth (i.e. what is r)?

$$P_{12} = (1+r)^{12} P_0 = (1+r)^{12} \text{ and } P_{12} = 4096$$

gives $(1+r)^{12} = 4096 \Rightarrow 1+r = \sqrt[12]{4096} = 2 \Rightarrow r = 1.$

The number r is usually converted to a percentage. $r=1$ means the population increases by 100% from generation to generation.

If we graphed this example it would look like



and the curve gets steeper and steeper.

For example, $P_{24} = 2^{24} = 16,777,216$

≈ 16.7 million infected

$P_{32} = 2^{32} \approx 4.3$ billion infected.

The geometric sum formula:

$$P_0 + RP_0 + R^2P_0 + \dots + R^{N-1}P_0 = \frac{(R^N - 1)P_0}{R-1}.$$

We can use this to calculate how much money we have after taking option #2 from the beginning of lecture.

In that situation, $P_0 = 1$ penny, $R = 2$,
and $N-1 = 31$, so

$$\begin{pmatrix}\text{total amount of} \\ \text{money after 31 days}\end{pmatrix} = 1 + 2 + 4 + 8 + \dots + 107374182$$

$$= P_0 + RP_0 + R^2P_0 + \dots + R^{N-1}P_0$$

$$= \frac{(R^N - 1)P_0}{R-1}$$

$$= \frac{(2^{31}-1)\times 1}{2-1} = 2^{31} - 1$$

$$= 2{,}147{,}483{,}647$$

So we would have

$21,474,836.47 after 31 days,

or around $21 million.