Ch. 9 (continued): More exponential growth; Logistic growth

Recall the very useful geometric sum formula:

$$P_0 + RP_0 + R^2P_0 + \dots + R^{N-1}P_0 = \frac{(R^N - 1)P_0}{R-1}$$

This is actually rather straightforward to derive. First note that on both sides of the equation, a P_0 can be factored out, so we need only show the following:

$$1 + R + R^2 + \dots + R^{N-1} = \frac{R^N - 1}{R-1}. \quad (\star)$$

Let $S_N = 1 + R + R^2 + \dots + R^{N-1}$. The goal is to show $S_N = \frac{R^N - 1}{R-1}$.

Note
$$\begin{aligned} RS_N = R(1 + R + \dots + R^{N-1}) &= R + R^2 + \dots + R^N \\ &= (1 + R + R^2 + \dots + R^N) - 1 \\ &= S_{N+1} - 1. \end{aligned}$$

Similarly, $RS_{N-1} = S_N - 1$.

Also,
$$\begin{aligned} S_N - S_{N-1} &= (1 + R + \dots + R^{N-1}) - (1 + R + \dots + R^{N-2}) \\ &= R^{N-1}. \end{aligned}$$

So we have the 2 equations

$$\begin{cases} RS_{N-1} = S_N - 1 \\ S_N - S_{N-1} = R^{N-1} \end{cases}$$

The 2nd equation gives $S_{N-1} = S_N - R^{N-1}$ and plugging this into the 1st equation yields

$$RS_{N-1} = S_N - 1$$

$$\Rightarrow R(S_N - R^{N-1}) = S_N - 1$$

$$\Rightarrow RS_N - R^N = S_N - 1$$

$$\Rightarrow (R-1)S_N = R^N - 1$$

$$\Rightarrow S_N = \frac{R^N - 1}{R-1}$$

which proves (★).

I will not ask you to prove this on the final exam, but understanding where the formula comes from should help with any manipulations you perform related to the geometric sum formula (and it's a very nice derivation, of course!).

Example A restaurant chain is increasing its per month profit every month by a factor of 3. In Jan. 2014, their profits were \$100,000. How much money did they make from Jan 2014 to Jan 2015?

$P_0 = 100{,}000$, $P_1 = 3 \times P_0 = 300{,}000$, $P_2 = 3^2 \times 100{,}000, \ldots$

$P_0 + RP_0 + R^2 P_0 + \cdots + R^{N-1} P_0$ (set $R = 3$, $P_0 = 100\text{k}$, $N-1 = 12$

$$= \left(\frac{R^N - 1}{R-1}\right) P_0 = \left(\frac{3^{13}-1}{3-1}\right)(100{,}000)$$

$$= 79{,}716{,}100{,}000$$

$$\simeq \$79.7 \text{ billion.}$$

What was their monthly profit in Jan 2015?

$$P_{12} = R^{12} P_0 = 3^{12} \times 100{,}000$$

$$= 53{,}144{,}100{,}000$$

$$\simeq \$53 \text{ billion.}$$

What was the monthly rate of growth?

$R = 1 + r$, $R = 3$, so $r = 2$, or 200%.

More precisely, the profits grew 200% every month.

Logistic Growth

This growth model takes into account a populations habitat — and the key feature that the model has built in is the principle that a population's growth rate is negatively impacted by the populations' density.

Instead of keeping track of the actual population size, it is more convenient here to only keep track of a population's p-value P_N at generation N. The p-value should be thought of as a percentage, and represents how much of the habitat that the population is taking up.
(The text uses the nice analogy of P_N being like the occupancy rate at a hotel.)

If we write C for the carrying capacity of the habitat (relative to the population) and P_N = population size of generation N, as usual, then

$$\text{p-value at generation } N = p_N = \frac{P_N}{C}.$$

(the p-value p_N is a lower case p.)

For example, $p_N = 0.75$ means that generation N is taking up 75% of the available habitat.

If we are told that the carrying capacity of the habitat is 100K, then we can solve for P_N:

$$p_N = 0.75 = \frac{P_N}{C} = \frac{P_N}{100{,}000} \Rightarrow P_N = 75{,}000$$

i.e. the N^{th} generation has 75,000 individuals.

The Logistic Equation:

$$\boxed{p_{N+1} = r(1-p_N)p_N}$$

for some constant r, called the growth parameter.

If the p-values of a population satisfy this sequence of equations for some fixed r, then we say that the population growth follows the logistic growth model.

Example: There is a pond whose carrying capacity for rainbow trout is 10,000 fish.

The growth rate for these trout is $2.5 = r$.

We start with 2000 rainbow trout.

Thus $p_0 = \frac{P_0}{C} = \frac{2000}{10{,}000} = 0.2$

Now to find p_1 we use the logistic equation:

$$p_1 = r(1-p_0)p_0 = 2.5(1-0.2)(0.2)$$
$$= 2.5(0.8)(0.2)$$
$$= 0.4$$

Thus $P_1 = C \cdot p_1 = 10{,}000 \times 0.4 = 4{,}000$

so the population has doubled.

We continue:

$$p_2 = 2.5(1-0.4)(0.4) = 0.6$$

so $P_2 = 6{,}000$, and

$$p_3 = 2.5(1-0.6)(0.6) = 0.6$$

so $P_3 = P_2 = 6{,}000$, and

$$p_4 = 2.5(1-0.6)(0.6) = p_3 = 0.6,$$

and we notice that the population has stablized at 6,000, or 60% of the carrying capacity.