MAT331 Exercises, Fall 2019

set number 1

NOTE: *Each exercise is worth 10 points and can be turned in at any time before its "expiration date". At the end of the semester, I will expect you to have turned in at least 1/2 of the exercises assigned. If you do more, I will pick your best grades. If you do less, the missing grades will be counted as zeros. Altogether, these will count the same as one project.*

Many of these problems will require you to use the help system and/or read the text to figure out what commands you need to use and how to use them.

1. (*expires 8/30*) Plot the function $f(x) = 2\sin x - x^3 - 1/5$, for $x \in [-4, 4]$. Find all the zeros of the function with an accuracy of 20 decimal digits. *Hint: See* `Digits`, `fsolve`.

2. (*expires 9/9*) If $p(x)$ is a complex polynomial with real coefficients, it is well known that it can be factored into a product of linear and quadratic terms with real coefficients, or into a product of linear terms only if the coefficients are allowed to be complex.

   First, use Maple to write $q(z) = x^5 - 3x^4 - 3x^3 + 9x^2 - 10x + 30$ as a product of three *exact* linear factors and one quadratic term, all with real coefficients. By exact, I mean you should leave any non-rational factors expressed as radicals; do not approximate terms like $\sqrt{3}$ as 1.73205, etc.

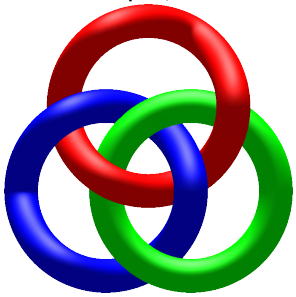
   Then write $q(x)$ as a product of only linear factors (which will involve complex numbers).
   Finally, do the same for product $p(x) = x^5 - 2x^4 - 10x^3 + 20x^2 - 16x + 32$.
   Hint: Note that this question asks for four different answers, two for each polynomial. While the maple command `factor` *is relevant, it will need a little assistance to be able to answer all four parts. See also* `RootOf` *and maybe* `convert,radical`*, or perhaps* `solve`*. Alternatively, there are other ways to do this. For example, using* `product` *or* `PolynomialTools`.

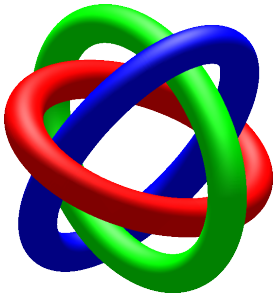
3. (*expires 9/9*) Draw a graph showing both $\cos(x)$ and its fifth Taylor polynomial (that is, $1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$) for x between -4 and 4. What degree of Taylor polynomial seems to be needed to get good agreement in this range? Think of a suitable way to demonstrate that the approximation you have taken is "good"– what is a good definition of "good" here? Be sure to **explain** your choice of the meaning of "good".
   Hint: Note that you can use `taylor` *to get the Taylor series for* $\cos(x)$*. The* `mtaylor` *command actually returns a polynomial, instead of a series with a* $O(x^n)$ *term; this could be helpful.*

4. (*expires 9/9*) The Borromean rings are a set of three mutually interlocked rings, arranged so that if one ring is removed, the other two become unlinked.

   Use Maple to make an image of the Borromean rings in $\mathbb{R}^3$. *Hint: I suggest using* `tubeplot` *from the* `plots` *package. The real challenge of this problem is figuring out how to parameterize the three curves.*

   Note that the Borromean rings can not be made from flat, round rings. Either some of the rings have to be bent a little to weave through each other, or flat rings which are elliptical can be used. See the figures below (made with Maple).





MAT331 Exercises, Fall 2019

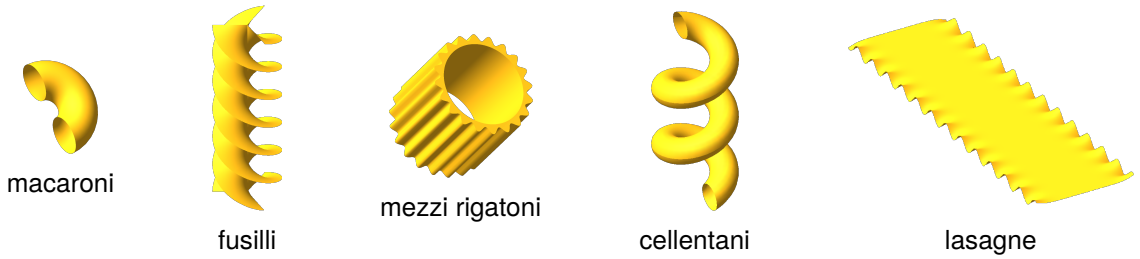
set number 2

5. (*expires 9/16*) Consider the planar curve γ defined by $x^2y^3 + y^2 + y - 2e^x = 0$. Using **only** Maple, find the slope of the tangent line to the curve at $(0, 1)$. Then plot the curve and the tangent line on the same graph.
   Hint: you might want to use `implicitplot` *and* `display` *from the* `plots` *library. You might find* `implicitdiff` *helpful, too.*

6. (*expires 9/16*) Define a Maple function g that, given a positive integer k yields the sum of the first k primes. What is k such that $g(k) \leq 100,000$ but $g(k+1) > 100,000$?
   You might find `ithprime` helpful, and probably `add` (rather than `sum`).

7. (*expires 9/16*) Write a function that, when given a positive integer n as input, will return the n^{th} digit of π (where 3 is viewed as the 0^{th} digit of π, and 4 is the 2^{nd} digit). What is the 2019^{th} digit of π?
   Using `floor` might be helpful, but you could have other ideas.

8. (*expires 9/16*) Use Maple to make pictures of the following pasta.



Here are some relevant equations, in no particular order.

$$z = \sin(2y)\left(1 - e^{-(x/6)^8}\right) \quad -6 \leq x \leq 6, \quad -20 \leq y \leq 20$$

$$\tau = 1 \quad 0 \leq \phi \leq \pi, \quad -\pi \leq \sigma \leq \pi \quad \text{(toroidal coordinates)}$$

$$x = \left(1 + \frac{\cos(s)}{2}\right)\cos(t) \quad y = \left(1 + \frac{\cos(s)}{2}\right)\sin(t) \quad z = 0.4t + \frac{\sin(s)}{2} \qquad \begin{array}{l} 0 \leq s \leq 2\pi \\ \frac{\pi}{2} \leq t \leq \frac{11\pi}{2} \end{array}$$

$$\left\{\begin{array}{lll} x = r\sin(t) & y = r\cos(t) & z = t/2 \\ x = r\sin\left(t + \frac{2\pi}{3}\right) & y = r\cos\left(t + \frac{2\pi}{3}\right) & z = t/2 \\ x = r\sin\left(t - \frac{2\pi}{3}\right) & y = r\cos\left(t - \frac{2\pi}{3}\right) & z = t/2 \end{array}\right\} \qquad \begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq t \leq 4\pi \end{array}$$

$$6 \leq r \leq 7 + \sin(20\theta)/2, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 14 \qquad \text{(cylindrical coordinates)}$$

To help you get started, the Maple worksheet called pasta.mw draws Mezzi Rigatoni. For full credit, your pasta should look like pasta, with appropriate coloring, viewpoint, smoothness, and lighting. Sauce is optional.

9. (*expires 9/23*) Given the set of points

$$(0,3),(1,0),(2,4),(3,4),(4,-1),(6,4),(7,0),(9,2),(10,10),$$

Find the polynomial of degree 8 that passes through all of them. If you wish, you can use `CurveFitting[PolynomialInterpolation]`, or you can calculate all of the relevant equations.

Then find the polynomial of degree 9 which passes through these points and also has a critical point at $x = 6$.

Also find the polynomial of degree 10 passing through the points with critical points at both $x = 1$ and $x = 6$.

Finally, make a plot displaying all three graphs, together with the data points. Be sure that your plot shows the data points clearly (as points, not connected lines), and clearly distinguishes the three functions. Including a legend (see `plot,options`) is one good way to do this.

To avoid typing in the points, you can load them from the web at `http://www.math.stonybrook.edu/~scott/mat331.fall19/problems/polydata.txt`, which defines a list called `polydata` containing them.

10. (*expires 9/23*) Using the same data as the previous problem, find the "natural" cubic spline which interpolates the data.

Also calculate the cubic spline which has derivative 0 at $x = 0$ and $x = 10$, and then make a plot showing the data points and these two curves on the same axes. (You will probably have to read the help page on Spline Continuity and End Conditions to see how to adjust the derivatives at the endpoints.)

11. (*expires 9/23*) Consider the set of six points (defined as `CSNY` on this page)

$$(-3.1,-1.94),(-2.2,3.82),(-0.95,-4.05),(0.8,-0.05),(1.3,-0.221),(4.7,-1.30).$$

Find the function of the form

$$f(x) = a\cos(x) + b\sin(x) + k\cos(2x) + d\sin(2x) + g\cos(3x) + h\sin(3x)$$

with appropriate values of the constants a, b, k, d, g, h (correct to at least 6 significant figures) so that $f(x)$ passes through the given points.

Then plot the points and $f(x)$ on the same set of axes for $-2\pi < x < 2\pi$.

12. (*expires 9/23*) Similar to the class discussion on Sept. 12, the worksheet sliderfit.mw contains an interactive slider to demonstrate how the interpolating polynomial changes when the y-value of one of the data points is moved.

Modify this worksheet to add another slider which allows the user to also move the x-value of the same point.

Also modify it to include the graph of the corresponding cubic spline in the plot.

Hint: you will need to modify the "startup code" at the top, and right-clicking on the slider to select **"Edit Value Changed Code..."** will be useful.

13. (*expires 9/30*) On the Wikipedia page on interpolation is an example of a cubic spline interpolating several points on an epitrochoid. An epitrochoid can be written in parametric form as $\{x(t) = (a+b)\cos(t) - c\,\cos\left((a/b+1)t\right),\ y(t) = (a+b)\sin(t) - c\,\sin\left((a/b+1)t\right)\}$. The file epidata.txt defines a list of points `epipts` on an epitrochoid, evaluated at each of the corresponding t-values given in the list `tvals`.

    Use `CurveFitting[Spline]` to determine the cubic splines interpolating the given points, and then produce a plot showing both the interpolating curve and the points, analogous to the one on Wikipedia. Plot your spline for $0 \le t \le 4\pi$.

    It is irrelevant for doing the problem, but the epitrochoid used has $a = 5$, $b = 2$, and $c = 5$.

14. (*expires 9/30*) Fit the points $(-1.9, -4.7), (-0.8, 1.2), (0.1, 2.8), (1.4, -1.2), (1.8, -3.5)$ with a quadratic function $f(x) = ax^2 + bx + c$, using the least square method.

    You can load these data points from the web via the link at fitquad.txt which defines a list `fitquad` containing them.

15. (*expires 9/30*) The file fitexp.txt defines a list `expdata` with 21 data points approximating an exponential curve of the form $y = ae^{bx}$.

    Find a and b by taking an appropriate logarithm, then use `CurveFitting[LeastSquares]` to find the resulting "best" line. Then transform this line appropriately to get an exponential curve which approximates the given data. The `map` command might be helpful.

    Plot the exponential and the points from `expdata` on the same axes, and write the approximating exponential in the form $y = ae^{bx}$.

16. (*expires 9/30*) Fit the set of points

    $$(1.021, -4.30), (1.001, -2.12), (0.99, 0.52), (1.03, 2.51), (1.00, 3.34), (1.02, 5.30)$$

    with a line, using the least square method. Plotting these points and the line on the same graph shows that this is not a good fit. (This is most apparent if you have a plot with, say, $0 < x < 2$ and $-5 < y < 6$.) Think of a better way to find a line which *is* a good fit and use Maple to do it. Explain in your solution why you think your better way is indeed an improvement. In case you don't want to retype them, the file badfit.txt defines a list `fitme` containing these points.

17. (*expires 9/30*) In this problem we will estimate the charge of the electron.

    If an electron of energy E is thrown into a magnetic field B which is perpendicular to its velocity, the electron will be deflected into a circular trajectory of radius r. The relation between these three quantities is:

    $$B\,r\,e = \frac{E^2}{m^2c^4}\sqrt{E^2 - m^2c^4}, \tag{1}$$

    where e and m are, respectively, the charge and the mass of the electron, and c is the speed of light ($2.9979 \times 10^8\ \frac{\text{meter}}{\text{sec}}$). The rest mass of the electron is defined by $E_0 = mc^2$, and is about equal to 8.817×10^{-14} Joules. In our experimental set-up the energy of the emitted electrons is set to be $E = 2.511E_0$.

    The file electron.txt defines a list called `electron`. Each element of the list is a pair of the form $[B_i, r_i]$; these quantities are expressed in Teslas and meters. Use least square fitting to determine the best value for e.

    Hint: Notice that the right hand side of eqn (1) is just a constant—calculate it once and for all and give it a name. Then eqn (1) becomes an equation which is linear in the unknown e. To verify your solution: $e \approx 1.602 \times 10^{-19}$ Coulomb. Physical constants courtesy of N.I.S.T.

18. (*expires 10/16*) Find an analytic expression for all the solutions to the differential equation

$$\frac{d}{dt}x(t) = -2x(t), \quad t \in \mathbb{R}.$$

Among those, write the formula for the one satisfying $x(0) = 3$.

Hint: This is simple enough that you can do this in your head. But I suggest at least trying to get Maple to do this for you. See the help page `HowDoI,SolveAnOrdinaryDifferentialEquation` for more information.

19. (*expires 10/16*) Have Maple find analytic solutions to the following system of differential equations,

$$\begin{cases} y''(t) - z(t) &= e^t, \\ z'(t) - y(t) &= 0, \end{cases}$$

with initial conditions: $y(0) = 1$, $y'(0) = 0$, $z(0) = k$. Let us denote the solutions by $y_k(t)$, $z_k(t)$ (since they depend on the parameter k).

For k taking all integer values from -10 to 10, and $t \in [-4, 2]$, plot the functions y_k in blue, and the functions z_k in red, all on the same graph. (Yes, you will then have 42 functions plotted on the same graph.)

This is certainly a case when you don't want to retype the functions that Maple finds. You will almost certainly need to read the help page mentioned above and/or the help on `dsolve`. I also found `subs`, `unapply`, and `seq` useful.

20. (*expires 10/16*) For the functions $y_k(t)$ and $z_k(t)$ found in the previous problem, plot the parametric curves $\varphi_k(t) = [y_k(t), z_k(t)]$ for integer values of k between -7 and 7 and $-10 < t < 4$ on the same graph. Use the `view` option to plot in order to only show what lies in the region $-15 < y < 15$, $-15 < z < 15$ (with y and z having the same scale). Use a sequence of colors so that each solution is a different color, and the coloring follows a predictable pattern. It would be nice to include a legend at the right of the plot.

Hint: you might find something like `seq(COLOR(HUE,i/16),k=-7..7)` useful for the coloring, and `seq(sprintf("k=%d",k), k=-7..7)` helpful in making the legend.

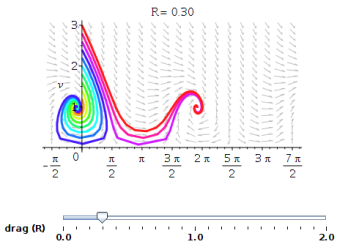
21. (*expires 10/16*) Find all the fixed points of the system

$$\begin{cases} \dot{x} &= x^2 + y, \\ \dot{y} &= x(y^2 - 1), \end{cases}$$

where a "fixed point" is a solution for which **both** $x(t)$ **and** $y(t)$ are constant. For each of these solutions you find, describe the behavior of the solutions that have initial conditions nearby. You can use Maple to figure out what happens for nearby points, or you can use more mathematical methods.

NOTE: The fact that there are various notations for differential equations is purely intentional.

22. (*expires 10/21*) On Oct. 8, we made an animation of solutions to the Phugoid model as the drag parameter R changes. Modify this to produce a plot of the corresponding picture in the (θ, v)-plane where the value of R is controlled by a slider, similar to Exercise 12. Your plot should change as the slider is moved, and look something like the image at right. You might want to refer to the worksheet sliderfit.mw.



23. (*expires 10/21*) In this problem will study the Lotke-Volterra predator-prey equations. In a very simple ecosystem there are two populations, whose numbers at a time t (with t in, say, years) are given by $f(t)$ (foxes) and $r(t)$ (rabbits). The evolution of these quantities obeys the system

$$\begin{cases} \dot{f}(t) &= G_f\, f(t) + E\, f(t)\, r(t), \\ \dot{r}(t) &= G_r\, r(t) - E\, f(t)\, r(t); \end{cases}$$

where G_f and G_r are the growth rates for the foxes and the rabbits, respectively, in the absence of each other. E is the probability of a fatal encounter between a fox and a rabbit (normalized per number of foxes and rabbits).

First, write some words to explain why these equations make sense. Then, fix $G_f = 0.4$, $G_r = 2.4$ (it's notorius that rabbits have the tendency to reproduce quickly) and $E = 0.01$. For a few initial conditions of your choice, plot the trajectories in the (f, r)-plane (say, with $0 \le f \le 1000$ and $0 \le r \le 1000$). For the same initial conditions, plot the actual solutions too (i.e, $f(t)$ against t, and $r(t)$ against t). Write some comments interpreting how the behavior of the solutions relates to what happens to the two species. (Here, to plot $f(t)$ against t, you can use the `scene` argument to `DEplot`, or you can use `dsolve` and maybe `plots[odeplot]`.)

Finally, repeat the same procedure with $G_f = -1.1$. Things change substantially. As above, explain how the solution behavior relates to the populations of foxes and rabbits. What does having $G_f = -1.1$ mean in the context of rabbit and fox populations?

24. (*expires 10/21*) Consider the differential equation corresponding to the the vector field

$$\mathbf{F}(x, y) = \left\langle -x(x^4 + y^4) - y,\ x - y(x^4 + y^4) \right\rangle\ .$$

Use Maple to draw the either the direction field or the vector field, together with some well-chosen solution curves. (I would use `DEplot` here, but you can use a combination of `fieldplot`, `dsolve` (with the `numeric` option), `plots[odeplot]`, and `plots[display]` if you prefer.)

Then *prove* that the origin is a global attractor in the future, i.e., for every solution $\mathbf{z}(t) = \big(x(t), y(t)\big)$, we have

$$\lim_{t \to +\infty} \mathbf{z}(t) = \mathbf{0}.$$

Note: The proof is not long, but requires a mathematical argument, not a maple calculation. The proof may depend on something you calculated in maple, but more will be needed. Polar coordinates can be your friend.

25. (*expires 10/28*) The (unforced) Van der Pol equation can be written as $x'' = \mu(1-x^2)x' - x$. This system was originally devised in the 1920s to describe behavior in electrical circuits that use vacuum tubes, but has many other applications including modeling the action potential of neurons. Letting $x' = y$ yields the system

$$\left\{ \frac{dx}{dt} = y, \quad \frac{dy}{dt} = \mu(1-x^2)y - x \right\} .$$

For all values of μ, the origin is a fixed point.

Show (by examining the linearization) that for $\mu = 0$ the origin is a center, but for $\mu > 0$ the origin is a source (a spiral source for $0 < \mu < 2$, although solutions near $(0,0)$ rotate clockwise for all $\mu > 0$).

However, for $\mu > 0$ solutions starting far away from the origin tend towards it, resulting in a *limit cycle* (this is a periodic solution which is not a fixed point, but corresponds to an oscillatory solution; nearby solutions tend towards it).

Use `DEplot` to illustrate the behavior of solutions for $\mu = 0$, $\mu = 0.5$, and $\mu = 2$, making plots in both the (x, y)-plane and for x as a function of t, choosing several initial conditions and coloring the corresponding solutions differently so they can be distinguished.

I suggest defining the system as a function of μ to minimize typing and typos. Looking at $-5 < x < 5$, $-5 < y < 5$ is a good range (when plotting x vs y, use the `scaling=constrained` option). A good selection of initial conditions has $x(0)$ ranging from -5.1 to 4.9 stepping by 1 and $y(0) = 0$, but let $-4 < t < 20$. Also using `obsrange=false` is a good idea here, and you should decrease the stepsize for more accurate solution curves. Using something like `display(⟨ DEplot(...) | DEplot(...,scene=[t,x]) ⟩)` is a good way to show the two plots side by side.
Maple will likely complain about encountering some singularities. If these messages annoy you, you can shut them up with `interface(warnlevel=0)`. Be aware this supresses all warning messages until you reset it (the default is `warnlevel=3`).

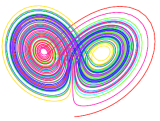
26. (*expires 10/28*) The Lorenz system is a system of three differential equations devised in the early 1960s as a model of convection in the atmosphere, and are written as

$$\left\{ \frac{dx}{dt} = \sigma(y-x), \quad \frac{dy}{dt} = x(\rho - z) - y, \quad \frac{dz}{dt} = xy - \beta z \right\} .$$

With the "classic" values $\sigma = 10$, $\beta = 8/3$, and $\rho = 28$, this system has three fixed points.

Find the three fixed points, and then compute the eigenvalues of the linearization of the system at each of them. What does this tell you about behavior of solutions near each of the fixed solutions?

27. (*expires 10/28*) The Lorenz equations given in Exercise 26 above give rise to a well-known chaotic attractor: almost all initial conditions result in solutions which are quickly attracted to a butterfly-like set, but these solutions are chaotic in the sense that they are aperiodic and have sensitive dependence to initial conditions. That is, arbitrarily small changes in initial conditions will cause the solutions to have drastically different futures.



The Maple command `DEplot3d` is similar to `DEplot`, but enables production of a 3-dimensional plot of solutions to a system of ODEs. Use `DEplot3d` to produce a three-dimensional plot of the Lorenz Attractor, using $\sigma = 10$, $\beta = 8/3$, and $\rho = 28$.

I suggest using $-30 < x < 30$, $-30 < y < 30$, and $0 < z < 52$ with `scaling=constrained`. I also used $0 < t < 75$ with `stepsize=0.01`, `method=dverk78`, `linecolor=t`, `thickness=1` to produce the image above, which is part of a single solution curve for the system with the color indicating the time. Your picture will differ a bit, depending on what initial condition you start with, but should retain the "butterfly" shape.

28. (*expires 11/21*) The text below was encrypted with a substitution cipher (that is, a permutation). Only the letters (both upper-case and lower-case) were substituted, leaving punctuation and spaces alone; the message has proper capitalization and punctuation. Determine the original message.

```
"wA'r aBeD WUeK AP XNaB NM U rALKNP USUeAJBMA NM zUM nPrB ZNAW U JUM ZWP'r
XBUeMNMO AP SXUD AWB aNPXNM." yWUA'r ZWUA rWB APXK AWB SPXNCB ZWBM rWB WUMKBK
AWBJ AWB BJSAD eBaPXaBe.

        lNCWUeK heULANOUM, "yWB zCUeXUAAN yNXA"
```

You can find the encrypted text on the class web page at `http://www.math.stonybrook.edu/~scott/mat331.fall19/problems/subscrypt.txt`. If you want to store this in a Maple string, you might want to either omit the quotation marks or prefix them with a backslash (i.e., \") when retyping them.

You do not need to write Maple that solves *every* such problem; the easiest way to do this problem involves some guesswork as well as knowledge of how English sentences are structured. After you get five or ten letters, the others should come easily.

You might find `CountCharacterOccurrences` or `CharacterFrequencies` from the library `StringTools` helpful. Depending on how you do things, `CharacterMap` could also be useful. Note that you don't really need to use Maple to do this problem, but of course you may.

29. (*expires 11/21*) In traditional "pen and paper" cryptography (that is, before the widespread availability of computers), messages were often written in uppercase letters with punctuation and spacing removed, then divided into blocks of five letters. For example, the start of this problem would be rendered as below.

```
INTRA DITIO NALPE NANDP APERC RYPTO GRAPH YTHAT ISBEF ORETH EWIDE SPREA DAVAI
LABIL ITYOF COMPU TERSM ESSAG ESWER EOFTE NWRIT TENIN UPPER CASEL ETTER SWITH
PUNCT UATIO NANDS PACIN GREMO VEDTH ENDIV IDEDI NTOBL OCKSO FFIVE LETTE RS
```

Write a Maple procedure to render a given text string this way.

Consider using `Select`, `UpperCase`, and `printf`. Also, it may be useful to remember that if the variable `msg` contains a string, one can refer to characters 10–14 of it with `msg[10..14]`.

30. (*expires 11/21*) In class on Nov. 12, we wrote code for a Caesar cipher (or see this worksheet). This doesn't work properly when there are characters in the plaintext that aren't part of the encryption alphabet. Fix this in two different ways:

- First, write a version that just removes these offending characters when encrypting.
- Then write a version that only shifts the characters that occur in the alphabet, and replaces any other characters with an underscore character (_).

31. (*expires 11/21*) The cryptography chapter in the notes is called "`fsqFsHn sGGousG`", which is actually the result of applying a Caesar cipher to its original title. A 53-character alphabet consisting of all the upper-case letters, a space, and then all the lower-case letters was used; consequently the space in the middle might or might not correspond to a space in the title. Determine what the original title was.

32. (*expires 11/29*) The text below was encrypted using a Vignère cipher on the 28-letter alphabet consisting of the standard English alphabet, a space, and a period in that order (that is, "`abcdefghijklmnopqrstuvwxyz .`"). The keyword is known to be four characters long. Decrypt the message given below, and determine the encrypting keyword. You can find Maple for the Vignère cipher in this worksheet, and the encrypted text (with spaces unchanged) in vignere4.txt.)

    The second two lines in the encrypted text begin with a space; linebreaks are not significant in the message. To make it easier to see the spaces, they have been written here as underscores (_).

    ```
    xvlfpbebud_ijxhw.tlxcq_wiv_ijgsw.guofpoi.wijii_fmjbbjh_.pj
    _gppnruptdm._dcduw.wijii_fmjbc.ihh.hefpcdbsjlh.dfbgzgkupcovs
    _ltpyrvpdr.cow.iaolpaepjtbgzgkupcovs.
    ```

    Hint: As in the earlier problems, the message follows the rules of ordinary English, so spaces are quite common. Periods come at the end of sentences, so while they are not common, this can be a useful clue.

33. (*expires 11/29*) Modify the implementation of the Vignère cipher as used in class (or in the worksheet Crypto.mw) to use the contents of a webpage for the keyphrase. That is, be able to encrypt some text with a command like `WebVignere(text,"http://www.math.stonybrook.edu/~scott/mat331.fall19/problems/goldbug.txt")`. Be careful to account for the situation where the webpage may contain characters not in the `Alphabet`, and give an error if the URL can't be read (for example, if you use `http://example.com/nosuchurl` as your key). You should find the commands in the `HTTP` package useful.

34. (*expires 11/29*) The string below was encrypted using an affine cipher on the 27-letter alphabet "` abcdefghijklmnopqrstuvwxyz`" (there is a space in the 0^{th} position.) Decrypt it.

    ```
    fmw segjaweoouanerj a ceyqrype aswaheoaqbrqabeafrua eeaojerf afmjeayperjpu
    ```

    Hint: this phrase follows the the typical pattern in English where there are (almost) as many spaces as words (and so spaces are very common), and the letter "e" is also very common. You can use the technique described in section 7 of the cryptography chapter of the notes. In particular, using `msolve` should be helpful, and maybe `CharacterFrequencies` if you, like me, don't count so well.

    If you wish, the encrypted text can be loaded from the file afftext.txt. A version of the the affine cipher can be found in the worksheet Crypto.mw, or you can write your own.

35. (*expires 11/29*) The encrypted text below is a quote widely attributed to Albert Einstein (but probably not actually said by him).

    ```
    EINsteinc+eoplWcaewXoy.wNRjUkOeQaQQDARUzypyrmaFdnhZSdr-RaUzxpOvXc,Lv,lNLSTEINein
    ```

    The encrypted text can be found in the file einsteincrypt.txt. It was enrcrypted using the `Affine` cipher from class (and from Crypto.mw), using an `Alphabet` which is 57 characters long, consisting of a space, the upper-case letters A-Z, a plus, the lower-case letters a-z, a comma, a period, and finally a hyphen. That is, (the first character is a space):

    ```
    ABCDEFGHIJKLMNOPQRSTUVWXYZ+abcdefghijklmnopqrstuvwxyz.,-
    ```

    The `Affine` cipher was used with blocks of eight letters at a time (that is, with `base=57`^{8}), and the name "Albert Einstein" appears in the plaintext (fortunately for you, it is in a convenient location, or this would be much harder). I believe `msolve` will be helpful. Note that "Albert Einstein" is 15 characters long, so you will have to guess one letter and the location of the name.

    Decrypt the quotation.

36. (*expires 12/4*) The difficulty of breaking a cipher can be increased by inserting some random characters (or noise) into a known part of the plain text in such a way that it will interact with the actual text (sometimes this insertion of randomness is called "salting the plaintext").

    As an example of this, recall that we discussed (on Nov. 21) that if our character set came from an alphabet of length n, we could represent blocks of k-characters as numbers base n^k (for example, in the 53-character alphabet consisting of a space, upper-case letters A-Z, and lower-case letters a-z, the word `Hi` is represented by $8 + 35 \cdot 53 = 1863$ if we use 2-character blocks).

    If we agree that the first character of each block will be randomly chosen (and just removed after we decrypt), breaking the encryption becomes much, much harder. Indeed, the difficulty decreases by a factor of the number of possible salt characters.

    Modify the encryption scheme in `Affine` from Crypto.mw so that it can encrypt and decrypt including salt as the first character of each block. You may either make a new procedure `SaltedAffine` or modify `Affine` to accept an optional parameter to insert (and remove) the salt, as you see fit.

    As an explicit example, let us encode the string `Zombie Apocalypse` using the 53-character alphabet and 3-character blocks (where the first character of each block is randomly chosen from the `Alphabet)`. In this example, the given string corresponds to the list of character codes

    $$[26, 41, 39, 28, 35, 31, 0, 1, 42, 41, 29, 27, 38, 51, 42, 45, 31]$$

    which, when grouped into pairs and with a random character added as salt, gives

    $$[116551, 80721, 88936, 2842, 117404, 77428, 145275, 128676, 1652]$$

    (your numbers may differ by up to 53 because of the salt); viewing these as 3-graphs including the salt, corresponds to `DZoBmbBieg AIpovcaBlyspsIe`. If we then encrypt this using the affine encryption $x \mapsto 12347x + 56890 \mod 53^3$, we get

    $$[67005, 136439, 32930, 12092, 28729, 121189, 97219, 4118, 57985]$$

    or `MsWQdvQlKHPDCLJeGqQfhkXAChT` (unless you used the same salt, your encryption will differ significantly, but both should decrypt just fine.)

    If you were able to make this work, you should be able to decrypt the string[1]

    `MWLiuWVckaMOpfHuWPgGuWlyQovqkwBwWV nLZYhrySYihTUSFqFJuGxEtvxCWNxPxstkPwkkxo`

    which was encrypted using salted pairs (that is, 3-character blocks including salt) from the 53-character alphabet above, and applying the affine mapping $x \mapsto 47x + 31415 \mod 53^3$. Decrypt the phrase.

[1] also available as zombiecrypt.txt so you don't have to worry about typing errors.